

16 Complexe functies

Voorkennis Complexe getallen

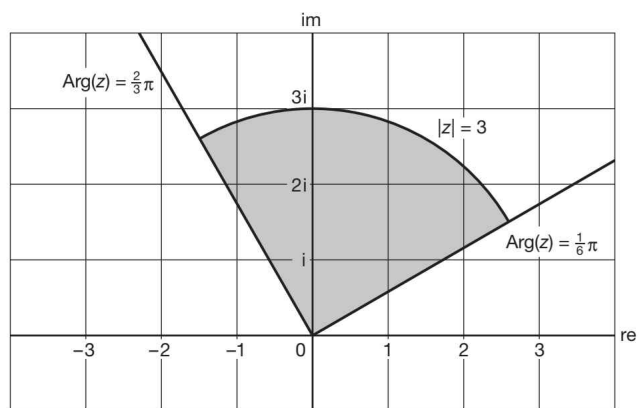
Bladzijde 154

- 1 a** $2\frac{1}{2}z - 2 - i = z + 1 + 5i$
 $1\frac{1}{2}z = 3 + 6i$
 $z = 2 + 4i$
- b** $1\frac{1}{4}z + 5 - 2i = z$
 $\frac{1}{4}z = -5 + 2i$
 $z = -20 + 8i$
- c** $(4 - i) \cdot z = 2 + i$
 $z = \frac{2 + i}{4 - i} \cdot \frac{4 + i}{4 + i} = \frac{8 + 6i + i^2}{16 - i^2} = \frac{8 + 6i - 1}{16 + 1} = \frac{7 + 6i}{17} = \frac{7}{17} + \frac{6}{17}i$
- d** $(3 + i) \cdot z - 1 + 2i = 5 + i$
 $(3 + i) \cdot z = 6 - i$
 $z = \frac{6 - i}{3 + i} \cdot \frac{3 - i}{3 - i} = \frac{18 - 9i + i^2}{9 - i^2} = \frac{18 - 9i - 1}{9 + 1} = \frac{17 - 9i}{10} = 1\frac{7}{10} - \frac{9}{10}i$
- e** $z^2 - 2z + 5 = 0$
 $(z - 1)^2 - 1 + 5 = 0$
 $(z - 1)^2 = -4$
 $(z - 1)^2 = 4i^2$
 $z - 1 = 2i \vee z - 1 = -2i$
 $z = 1 + 2i \vee z = 1 - 2i$
- f** $z^2 - 5z + 10 = 0$
 $(z - 2\frac{1}{2})^2 - 6\frac{1}{4} + 10 = 0$
 $(z - 2\frac{1}{2})^2 = -3\frac{3}{4}$
 $(z - 2\frac{1}{2})^2 = \frac{15}{4}i^2$
 $z - 2\frac{1}{2} = \frac{1}{2}i\sqrt{15} \vee z - 2\frac{1}{2} = -\frac{1}{2}i\sqrt{15}$
 $z = 2\frac{1}{2} + \frac{1}{2}i\sqrt{15} \vee z = 2\frac{1}{2} - \frac{1}{2}i\sqrt{15}$

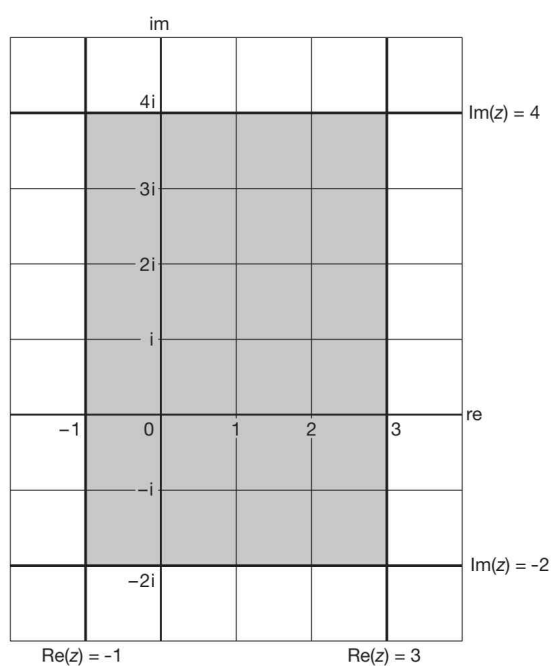
Bladzijde 155

- 2 a** $|3 - i| = \sqrt{3^2 + (-1)^2} = \sqrt{9 + 1} = \sqrt{10}$
 $\text{Arg}(3 - i) \approx -0,32$
- b** $|-4 + 3i| = \sqrt{(-4)^2 + 3^2} = \sqrt{16 + 9} = \sqrt{25} = 5$
 $\text{Arg}(-4 + 3i) \approx 2,50$
- c** $|5 - 2i| = \sqrt{5^2 + (-2)^2} = \sqrt{25 + 4} = \sqrt{29}$
 $\text{Arg}(5 - 2i) \approx -0,38$
- d** $|-6 - 8i| = \sqrt{(-6)^2 + (-8)^2} = \sqrt{36 + 64} = \sqrt{100} = 10$
 $\text{Arg}(-6 - 8i) \approx -2,21$
- 3 a** $|\sqrt{3} + i| = \sqrt{(\sqrt{3})^2 + 1^2} = \sqrt{3 + 1} = \sqrt{4} = 2$
 $\text{Arg}(\sqrt{3} + i) = \frac{1}{6}\pi$
- b** $|5 - 5i| = \sqrt{5^2 + (-5)^2} = \sqrt{25 + 25} = \sqrt{50} = 5\sqrt{2}$
 $\text{Arg}(5 - 5i) = -\frac{1}{4}\pi$
- c** $|-2 + 2i\sqrt{3}| = \sqrt{(-2)^2 + (2\sqrt{3})^2} = \sqrt{4 + 12} = \sqrt{16} = 4$
 $\text{Arg}(-2 + 2i\sqrt{3}) = \frac{2}{3}\pi$
- d** $|-3 - 3i\sqrt{3}| = \sqrt{(-3)^2 + (-3\sqrt{3})^2} = \sqrt{9 + 27} = \sqrt{36} = 6$
 $\text{Arg}(-3 - 3i\sqrt{3}) = -\frac{2}{3}\pi$

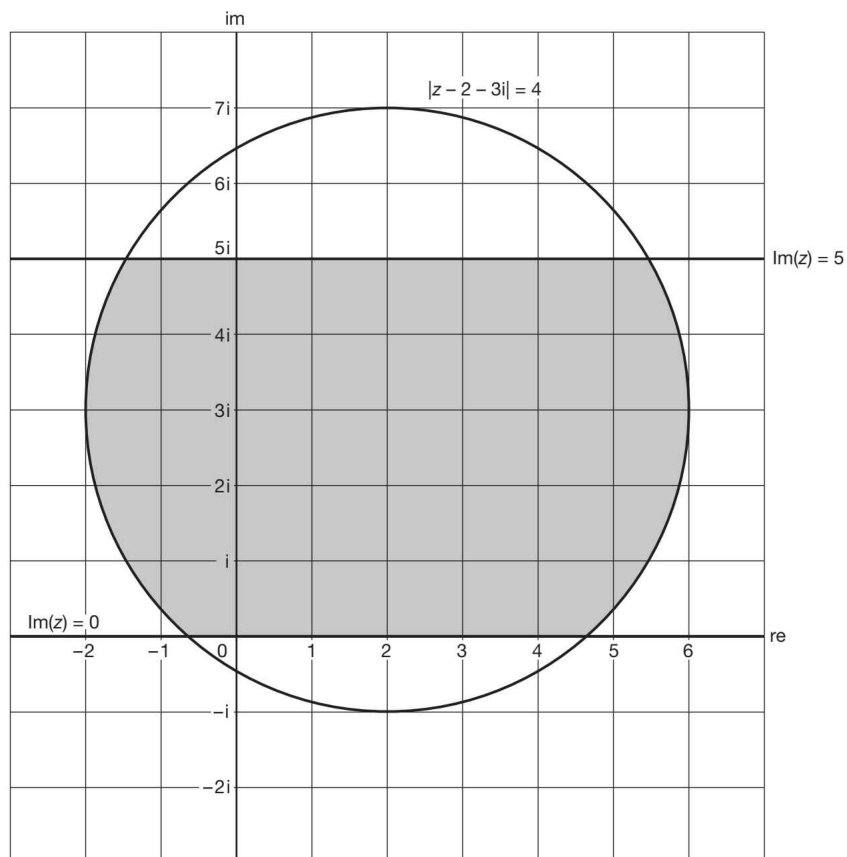
4 a



b

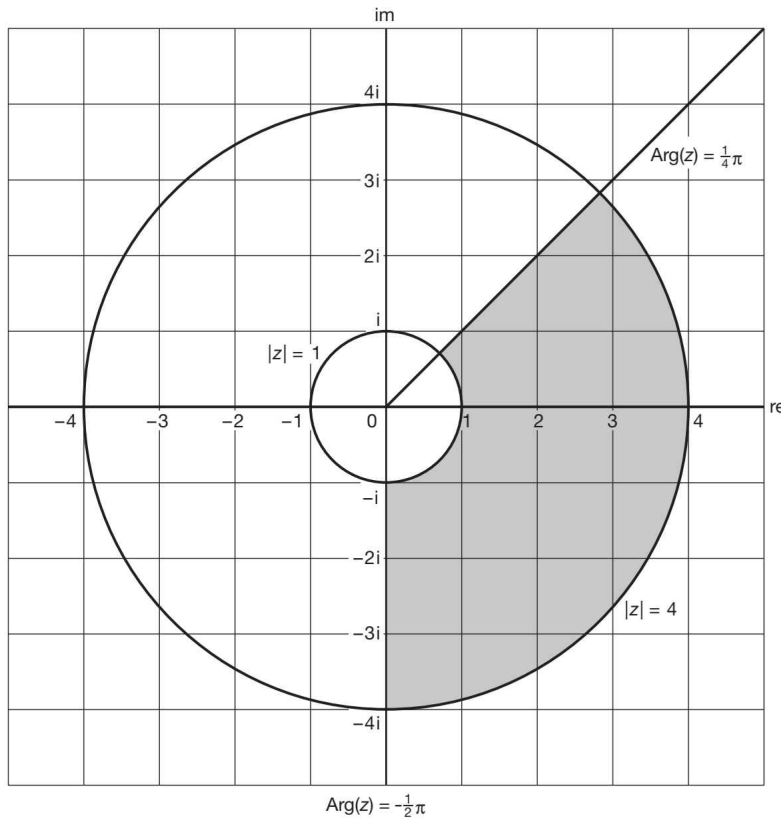


c



16

d

**Bladzijde 156**

- 5 a $|z| = \sqrt{3^2 + 2^2} = \sqrt{13}$ en $\arg(z) \approx 0,59$ geeft $z = \sqrt{13}e^{0,59i}$
 b $|z| = \sqrt{3^2 + (-4)^2} = 5$ en $\arg(z) \approx -0,93$ geeft $z = 5e^{-0,93i}$
 c $|z| = \sqrt{(\sqrt{3})^2 + (\sqrt{2})^2} = \sqrt{5}$ en $\arg(z) \approx 0,68$ geeft $z = \sqrt{5}e^{0,68i}$

Bladzijde 157

- 6 a $z = 3e^{-i} = 3(\cos(-1) + i\sin(-1)) = 3\cos(-1) + 3i\sin(-1) \approx 1,62 - 2,52i$
 b $z = \pi^2 e^{2i} = \pi^2(\cos(2) + i\sin(2)) = \pi^2\cos(2) + \pi^2 i\sin(2) \approx -4,11 + 8,97i$
 c $z = \sqrt{3}e^{\frac{1}{5}\pi i} = \sqrt{3}(\cos(\frac{1}{5}\pi) + i\sin(\frac{1}{5}\pi)) = \sqrt{3}\cos(\frac{1}{5}\pi) + i\sqrt{3}\sin(\frac{1}{5}\pi) \approx 1,40 + 1,02i$
- 7 a $z = 2e^{\frac{1}{4}\pi i} = 2(\cos(\frac{1}{4}\pi) + i\sin(\frac{1}{4}\pi)) = 2(\frac{1}{2}\sqrt{2} + \frac{1}{2}i\sqrt{2}) = \sqrt{2} + i\sqrt{2}$
 b $z = \sqrt{3}e^{\frac{1}{3}\pi i} = \sqrt{3}(\cos(\frac{1}{3}\pi) + i\sin(\frac{1}{3}\pi)) = \sqrt{3}(\frac{1}{2} + \frac{1}{2}i\sqrt{3}) = \frac{1}{2}\sqrt{3} + \frac{1}{2}i$
 c $z = 10e^{-\frac{1}{6}\pi i} = 10(\cos(-\frac{1}{6}\pi) + i\sin(-\frac{1}{6}\pi)) = 10(\frac{1}{2}\sqrt{3} - \frac{1}{2}i) = 5\sqrt{3} - 5i$
- 8 a $z = (1 - i)^5 = (\sqrt{2}e^{-\frac{1}{4}\pi i})^5 = 4\sqrt{2}e^{-\frac{5}{4}\pi i} = 4\sqrt{2}e^{\frac{3}{4}\pi i}$
 b $z = (1 - i\sqrt{3})^{-4} = (2e^{-\frac{1}{3}\pi i})^{-4} = \frac{1}{16}e^{\frac{4}{3}\pi i} = \frac{1}{16}e^{-\frac{2}{3}\pi i}$
 c $z = \frac{2 + 2i\sqrt{3}}{1 - i\sqrt{3}} = \frac{4e^{\frac{1}{3}\pi i}}{2e^{-\frac{1}{3}\pi i}} = 2e^{\frac{2}{3}\pi i}$
- 9 a $(z - 3)^2 = 1 + i\sqrt{3}$
 $(z - 3)^2 = 2e^{\frac{1}{3}\pi i + k \cdot 2\pi i}$
 $z - 3 = \sqrt{2}e^{\frac{1}{6}\pi i + k \cdot \pi i}$
 $z - 3 = \sqrt{2}e^{\frac{1}{6}\pi i} \vee z - 3 = \sqrt{2}e^{\frac{7}{6}\pi i}$
 $z - 3 = \frac{1}{2}\sqrt{6} + \frac{1}{2}i\sqrt{2} \vee z - 3 = -\frac{1}{2}\sqrt{6} - \frac{1}{2}i\sqrt{2}$
 $z = 3 + \frac{1}{2}\sqrt{6} + \frac{1}{2}i\sqrt{2} \vee z = 3 - \frac{1}{2}\sqrt{6} - \frac{1}{2}i\sqrt{2}$
 b $(z - 2 + i)^3 = -8i$
 $(z - 2 + i)^3 = 8e^{-\frac{1}{2}\pi i + k \cdot 2\pi i}$
 $z - 2 + i = 2e^{-\frac{1}{6}\pi i + k \cdot \frac{2}{3}\pi i}$
 $z - 2 + i = 2e^{-\frac{1}{6}\pi i} \vee z - 2 + i = 2e^{\frac{1}{2}\pi i} \vee z - 2 + i = 2e^{\frac{5}{6}\pi i}$
 $z - 2 + i = \sqrt{3} - i \vee z - 2 + i = 2i \vee z - 2 + i = -\sqrt{3} - i$
 $z = 2 + \sqrt{3} - 2i \vee z = 2 + i \vee z = 2 - \sqrt{3} - 2i$

$$c \quad z^4 = \frac{81}{16}$$

$$z^4 = \frac{81}{16} e^{k \cdot 2\pi i}$$

$$z = 1\frac{1}{2} e^{k \cdot \frac{1}{2}\pi i}$$

$$z = 1\frac{1}{2} \vee z = 1\frac{1}{2} e^{\frac{1}{2}\pi i} \vee z = 1\frac{1}{2} e^{\pi i} \vee z = 1\frac{1}{2} e^{\frac{3}{2}\pi i}$$

$$z = 1\frac{1}{2} \vee z = 1\frac{1}{2} i \vee z = -1\frac{1}{2} \vee z = -1\frac{1}{2} i$$

$$10 \quad a \quad \sqrt{-10i}, \text{ dus los op } z^2 = -10i$$

$$z^2 = 10 e^{-\frac{1}{2}\pi i + k \cdot 2\pi i}$$

$$z = \sqrt{10} e^{-\frac{1}{4}\pi i + k \cdot \pi i}$$

$$z = \sqrt{10} e^{-\frac{1}{4}\pi i} \vee z = \sqrt{10} e^{\frac{3}{4}\pi i}$$

$$z = \sqrt{5} - i\sqrt{5} \vee z = -\sqrt{5} + i\sqrt{5}$$

$$\text{Dus } \sqrt{-10i} = \sqrt{5} - i\sqrt{5} \vee \sqrt{-10i} = -\sqrt{5} + i\sqrt{5}.$$

$$b \quad -\frac{1}{8i} = i^2 \cdot \frac{1}{8i} = \frac{1}{8} i$$

$$\sqrt[3]{-\frac{1}{8i}}, \text{ dus los op } z^3 = -\frac{1}{8i}$$

$$z^3 = \frac{1}{8} i$$

$$z^3 = \frac{1}{8} e^{\frac{1}{2}\pi i + k \cdot 2\pi i}$$

$$z = \frac{1}{2} e^{\frac{1}{6}\pi i + k \cdot \frac{2}{3}\pi i}$$

$$z = \frac{1}{2} e^{\frac{1}{6}\pi i} \vee z = \frac{1}{2} e^{\frac{5}{6}\pi i} \vee z = \frac{1}{2} e^{\frac{9}{6}\pi i}$$

$$z = \frac{1}{4}\sqrt{3} + \frac{1}{4}i \vee z = -\frac{1}{4}\sqrt{3} + \frac{1}{4}i \vee z = -\frac{1}{2}i$$

$$\text{Dus } \sqrt[3]{-\frac{1}{8i}} = \frac{1}{4}\sqrt{3} + \frac{1}{4}i \vee \sqrt[3]{-\frac{1}{8i}} = -\frac{1}{4}\sqrt{3} + \frac{1}{4}i \vee \sqrt[3]{-\frac{1}{8i}} = -\frac{1}{2}i.$$

$$c \quad \sqrt[4]{-2 + 2i\sqrt{3}}, \text{ dus los op } z^4 = -2 + 2i\sqrt{3}$$

$$z^4 = 16 e^{\frac{2}{3}\pi i + k \cdot 2\pi i}$$

$$z = 2 e^{\frac{1}{6}\pi i + k \cdot \frac{1}{2}\pi i}$$

$$z = 2 e^{\frac{1}{6}\pi i} \vee z = 2 e^{\frac{2}{3}\pi i} \vee z = 2 e^{\frac{5}{6}\pi i} \vee z = 2 e^{\frac{7}{6}\pi i}$$

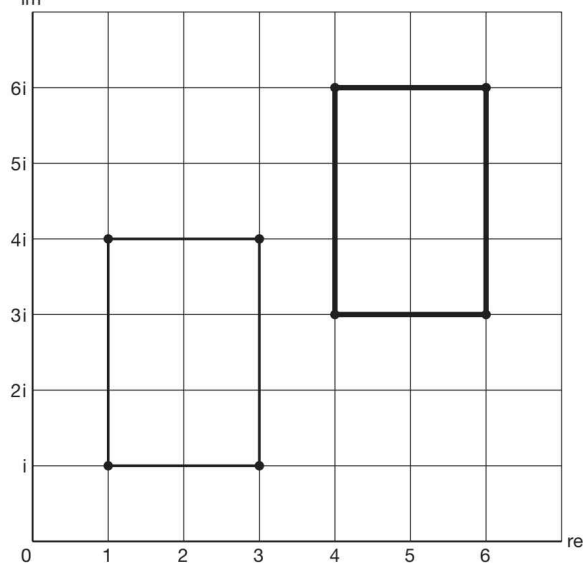
$$z = \sqrt{3} + i \vee z = -1 + i\sqrt{3} \vee z = -\sqrt{3} - i \vee z = 1 - i\sqrt{3}$$

$$\text{Dus } \sqrt[4]{-2 + 2i\sqrt{3}} = \sqrt{3} + i \vee \sqrt[4]{-2 + 2i\sqrt{3}} = -1 + i\sqrt{3} \vee \sqrt[4]{-2 + 2i\sqrt{3}} = -\sqrt{3} - i \vee \sqrt[4]{-2 + 2i\sqrt{3}} = 1 - i\sqrt{3}.$$

16.1 Lineaire complexe functies

Bladzijde 158

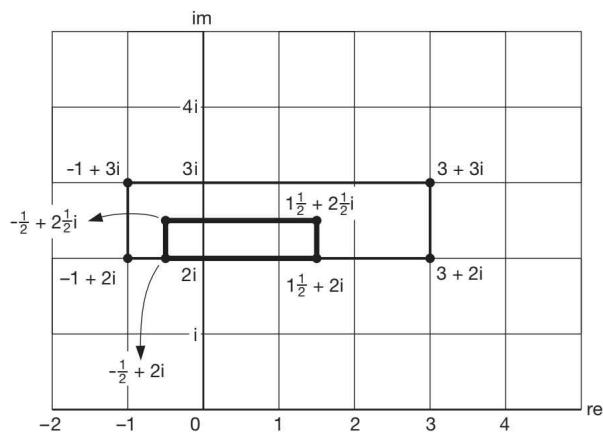
$$1 \quad a \quad \text{im}$$



- b $(1 + i) + (3 + 2i) = 4 + 3i$
 $(3 + i) + (3 + 2i) = 6 + 3i$
 $(3 + 4i) + (3 + 2i) = 6 + 6i$
 $(1 + 4i) + (3 + 2i) = 4 + 6i$
- c Zie de figuur van vraag a.
 De rechthoek is 3 naar rechts en 2 omhoog verschoven.

Bladzijde 160

- 2 a Bij $f(z) = \frac{1}{2}z + i$ hoort de vermenigvuldiging ten opzichte van 0 met $\frac{1}{2}$ gevolgd door de translatie $(0, 1)$.



- b $f(z) = 0$ geeft $\frac{1}{2}z + i = 0$

$$\frac{1}{2}z = -i$$

$$z = -2i$$

Het nulpunt is $-2i$.

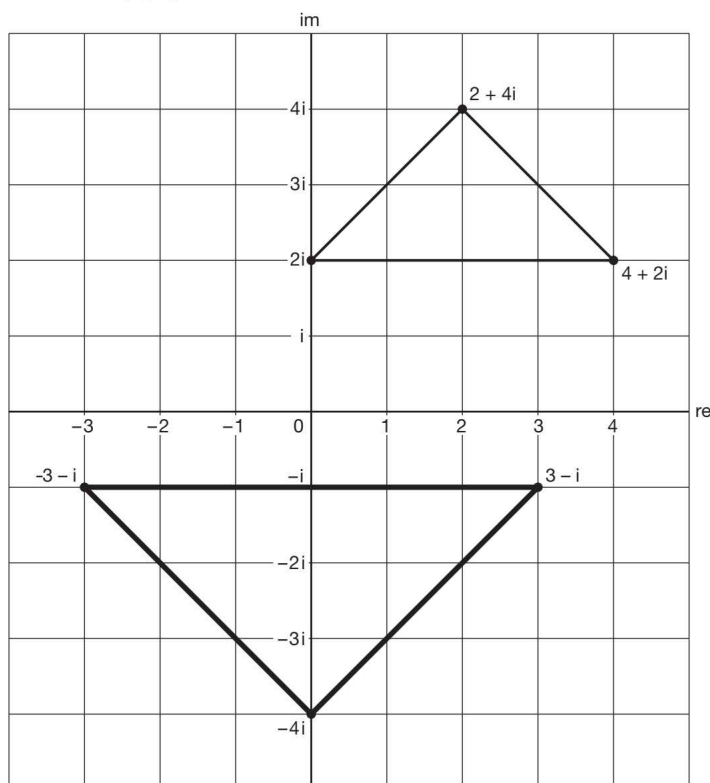
- c $f(z) = z$ geeft $\frac{1}{2}z + i = z$

$$-\frac{1}{2}z = -i$$

$$z = 2i$$

Het dekpunt is $2i$.

- 3 a Bij $f(z) = -\frac{1}{2}z + 3 + 2i$ hoort de vermenigvuldiging ten opzichte van 0 met $-\frac{1}{2}$ gevolgd door de translatie $(3, 2)$.



b $f(z) = 0$ geeft $-1\frac{1}{2}z + 3 + 2i = 0$

$$-1\frac{1}{2}z = -3 - 2i$$

$$z = 2 + 1\frac{1}{3}i$$

Het nulpunt is $2 + 1\frac{1}{3}i$.

c $f(z) = z$ geeft $-1\frac{1}{2}z + 3 + 2i = z$

$$-2\frac{1}{2}z = -3 - 2i$$

$$z = 1\frac{1}{5} + \frac{4}{5}i$$

Het dekpunt is $1\frac{1}{5} + \frac{4}{5}i$.

4 a $f(z) = 0$ geeft $3z + 2 - 4i = 0$

$$3z = -2 + 4i$$

$$z = -\frac{2}{3} + 1\frac{1}{3}i$$

Het nulpunt is $-\frac{2}{3} + 1\frac{1}{3}i$.

b $g(z) = 0$ geeft $\frac{1}{3}z + 5 = 0$

$$\frac{1}{3}z = -5$$

$$z = -15$$

Het nulpunt is -15 .

$f(z) = z$ geeft $3z + 2 - 4i = z$

$$2z = -2 + 4i$$

$$z = -1 + 2i$$

Het dekpunt is $-1 + 2i$.

$g(z) = z$ geeft $\frac{1}{3}z + 5 = z$

$$-\frac{2}{3}z = -5$$

$$z = 7\frac{1}{2}$$

Het dekpunt is $7\frac{1}{2}$.

5 a $f(z) = z$ geeft $az + 5 - 2i = z$

$$az - z = -5 + 2i$$

$$z(a - 1) = -5 + 2i$$

$$z = \frac{-5 + 2i}{a - 1}$$

Voor $a = 1$ heeft de functie f geen dekpunt.

b $f(z) = 0$ geeft $az + 5 - 2i = 0$

$$az = -5 + 2i$$

$$z = \frac{-5 + 2i}{a}$$

Voor $a = 0$ heeft de functie f geen nulpunt.

6 a $f(1 + 2i) = 0$ geeft $3(1 + 2i) + a + bi = 0$

$$3 + 6i + a + bi = 0$$

$$a + bi = -3 - 6i$$

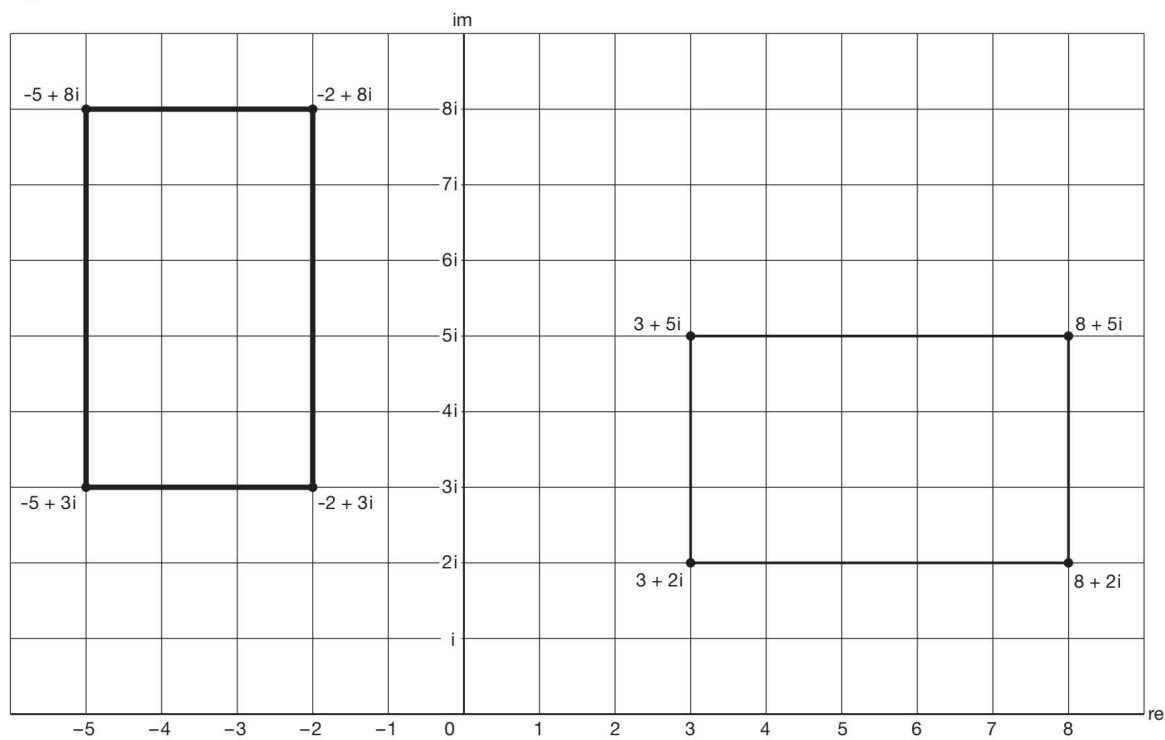
Dus $a = -3$ en $b = -6$.

b $f(1 + 2i) = 1 + 2i$ geeft $3(1 + 2i) + a + bi = 1 + 2i$

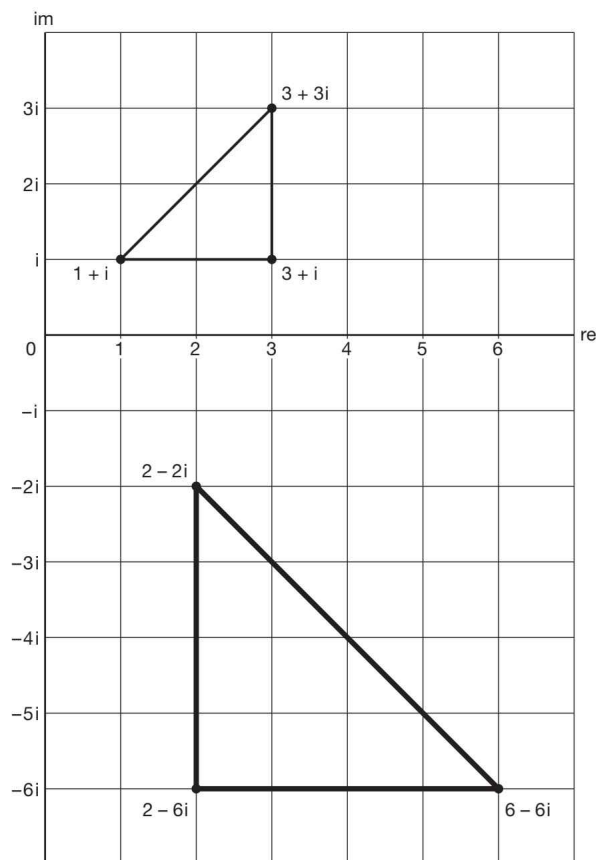
$$3 + 6i + a + bi = 1 + 2i$$

$$a + bi = -2 - 4i$$

Dus $a = -2$ en $b = -4$.

Bladzijde 161**7 a,b**

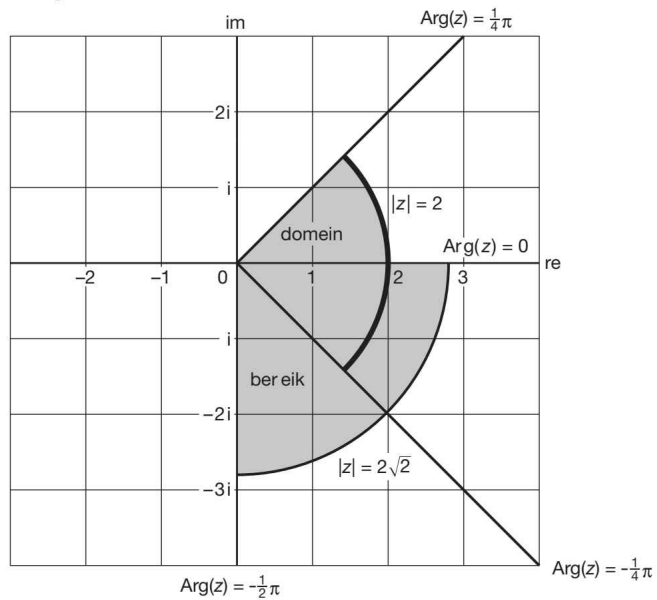
- b** $i(3 + 2i) = 3i - 2 = -2 + 3i$
 $i(8 + 2i) = 8i - 2 = -2 + 8i$
 $i(8 + 5i) = 8i - 5 = -5 + 8i$
 $i(3 + 5i) = 3i - 5 = -5 + 3i$
c Rotatie om 0 over $\frac{1}{2}\pi$.
d Rotatie om 0 over $\arg\left(\frac{1}{2} + \frac{1}{2}i\sqrt{3}\right) = \frac{1}{3}\pi$.

8 a,b

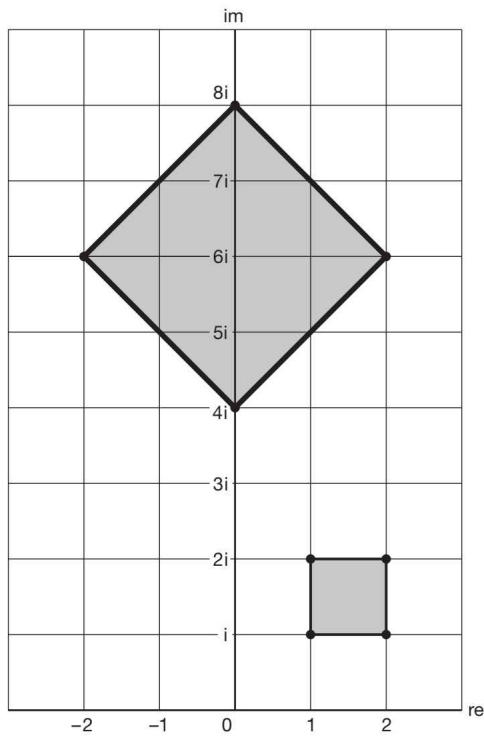
- b** $-2i(1 + i) = -2i + 2 = 2 - 2i$
 $-2i(3 + i) = -6i + 2 = 2 - 6i$
 $-2i(3 + 3i) = -6i + 6 = 6 - 6i$
c De rotatie om 0 over $-\frac{1}{2}\pi$.
 De vermenigvuldiging ten opzichte van 0 met 2.

Bladzijde 162

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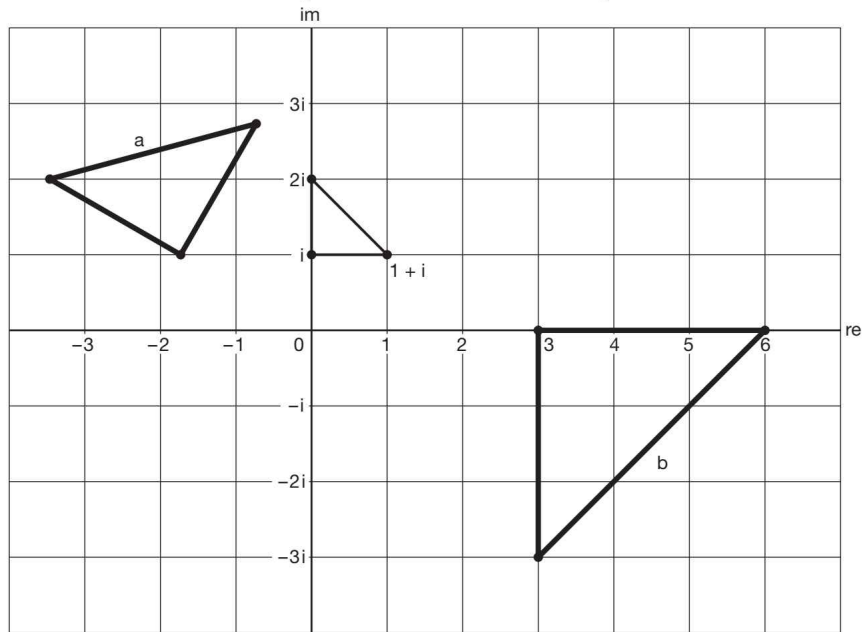


- 10 a $|2 + 2i| = 2\sqrt{2}$ en $\text{Arg}(2 + 2i) = \frac{1}{4}\pi$, dus rotatie om 0 over $\frac{1}{4}\pi$ en vermenigvuldiging ten opzichte van 0 met $2\sqrt{2}$.

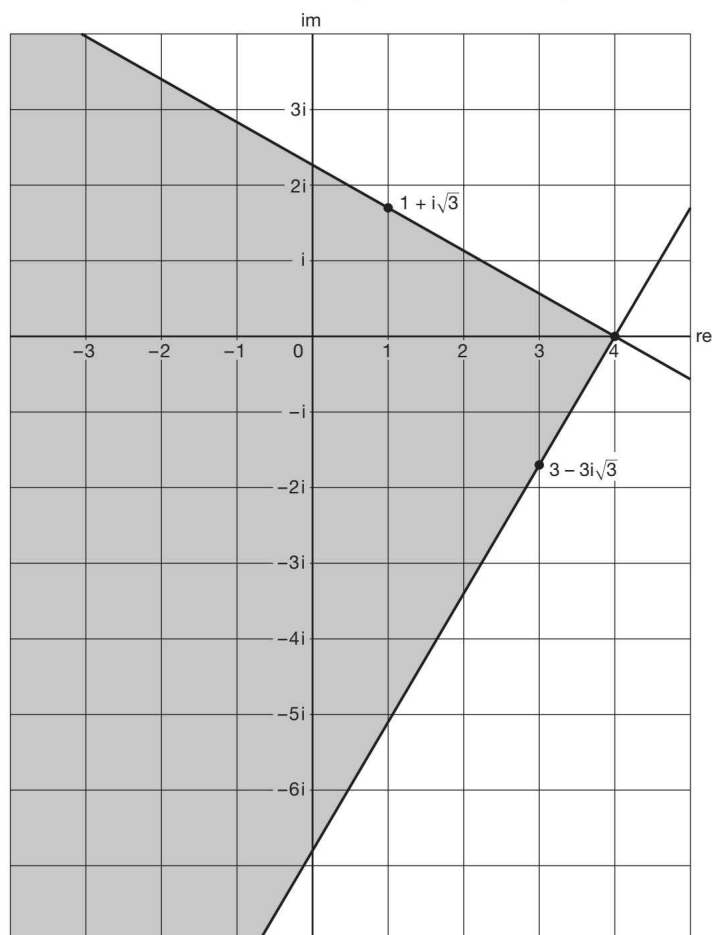


- b Het bereik is $2\sqrt{2} \leq |z| \leq 4\sqrt{2} \wedge \frac{5}{12}\pi \leq \text{Arg}(z) \leq \frac{7}{12}\pi$.

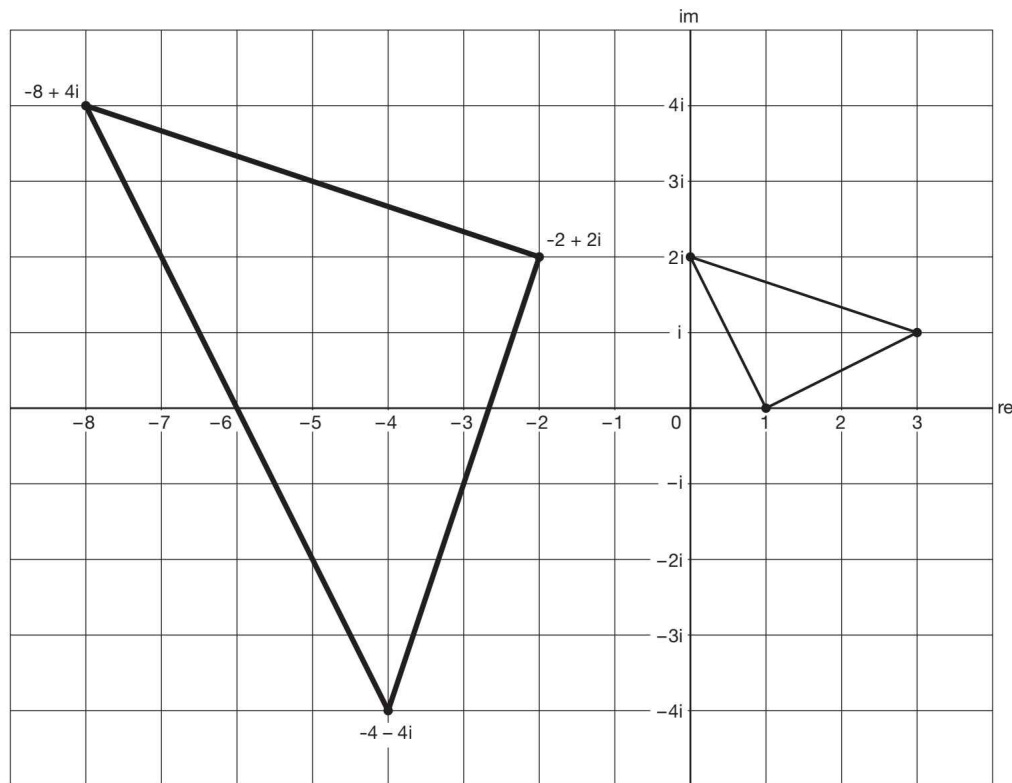
- 11 a,b** $|1 + i\sqrt{3}| = 2$ en $\text{Arg}(1 + i\sqrt{3}) = \frac{1}{3}\pi$, dus rotatie om 0 over $\frac{1}{3}\pi$ en vermenigvuldiging ten opzichte van 0 met 2.
 $|-3i| = 3$ en $\text{Arg}(-3i) = -\frac{1}{2}\pi$, dus rotatie om 0 over $-\frac{1}{2}\pi$ en vermenigvuldiging ten opzichte van 0 met 3.



- 12 a** $|\sqrt{3} - i| = 2$ en $\text{Arg}(\sqrt{3} - i) = -\frac{1}{6}\pi$, dus het bereik is $20 \leq z \leq 40 \wedge \frac{1}{3}\pi \leq \text{Arg}(z) \leq \frac{5}{6}\pi$.
b Het bereik is $z \geq 6 \wedge -\frac{1}{6}\pi \leq \text{Arg}(z) \leq \frac{1}{3}\pi$.
c De lijnen $\text{Re}(z) = \sqrt{3}$ en $\text{Im}(z) = 1$ snijden elkaar in $\sqrt{3} + i$.
 $f(\sqrt{3} + i) = (\sqrt{3} - i)(\sqrt{3} + i) = 3 - i^2 = 3 + 1 = 4$
 $\sqrt{3}$ ligt op $\text{Re}(z) = \sqrt{3}$.
 $f(\sqrt{3}) = (\sqrt{3} - i) \cdot \sqrt{3} = 3 - i\sqrt{3}$
 Het beeld van $\text{Re}(z) = \sqrt{3}$ is de lijn door 4 en $3 - i\sqrt{3}$.
 i ligt op $\text{Im}(z) = 1$.
 $f(i) = (\sqrt{3} - i) \cdot i = i\sqrt{3} - i^2 = 1 + i\sqrt{3}$
 Het beeld van $\text{Im}(z) = 1$ is de lijn door 4 en $1 + i\sqrt{3}$.



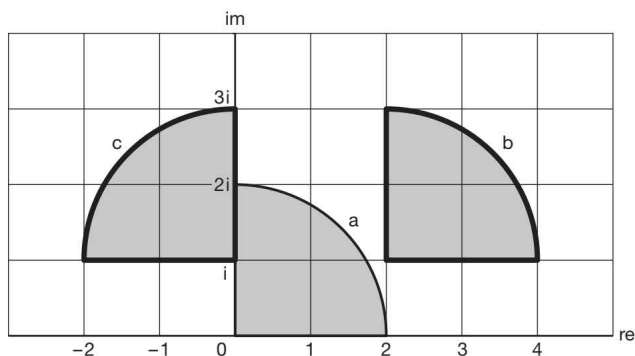
- 13 a** $|-2 + 2i| = 2\sqrt{2}$ en $\text{Arg}(-2 + 2i) = \frac{3}{4}\pi$, dus rotatie om 0 over $\frac{3}{4}\pi$ en vermenigvuldiging ten opzichte van 0 met $2\sqrt{2}$.
- $f(1) = (-2 + 2i) \cdot 1 = -2 + 2i$
 $f(3 + i) = (-2 + 2i) \cdot (3 + i) = -8 + 4i$
 $f(2i) = (-2 + 2i) \cdot 2i = -4 - 4i$



- b** $f(z) = 3 + 2i$ geeft $(-2 + 2i)z = 3 + 2i$
- $$z = \frac{3 + 2i}{-2 + 2i} \cdot \frac{-2 - 2i}{-2 - 2i} = \frac{-6 - 10i - 4i^2}{4 - 4i^2} = \frac{-2 - 10i}{8} = -\frac{1}{4} - 1\frac{1}{4}i$$
- c** $f(z) = f\left(\frac{1}{z}\right)$ geeft $(-2 + 2i)z = (-2 + 2i) \cdot \frac{1}{z}$
- $$z = \frac{1}{z}$$
- $$z^2 = 1$$
- $$z = 1 \vee z = -1$$

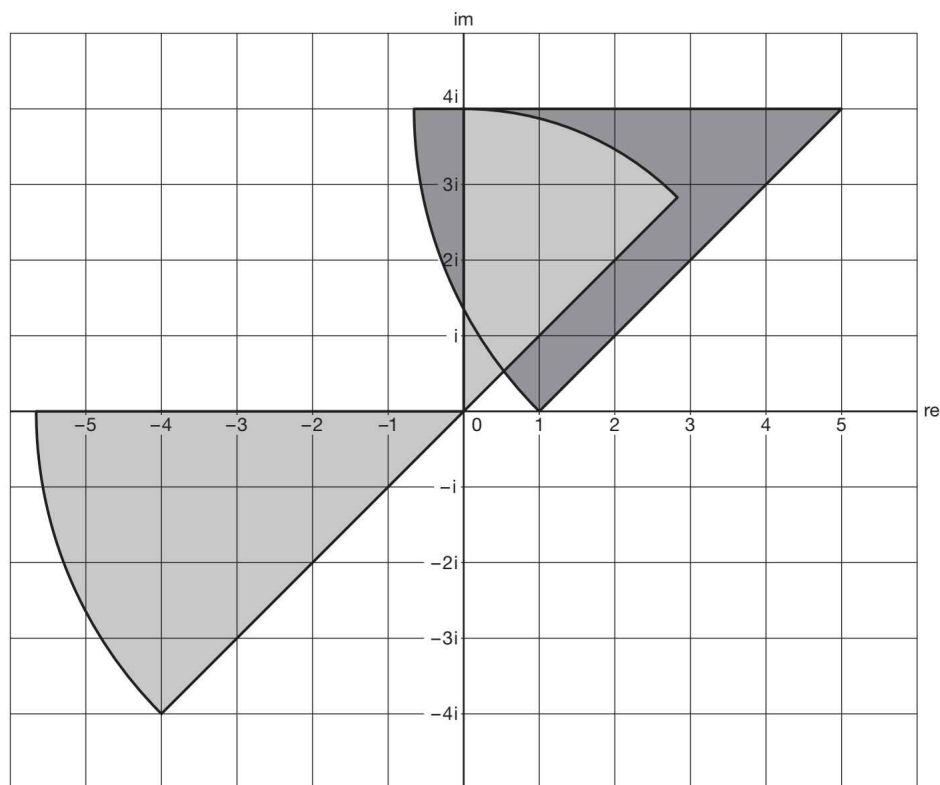
Bladzijde 163

- 14 a,b,c**

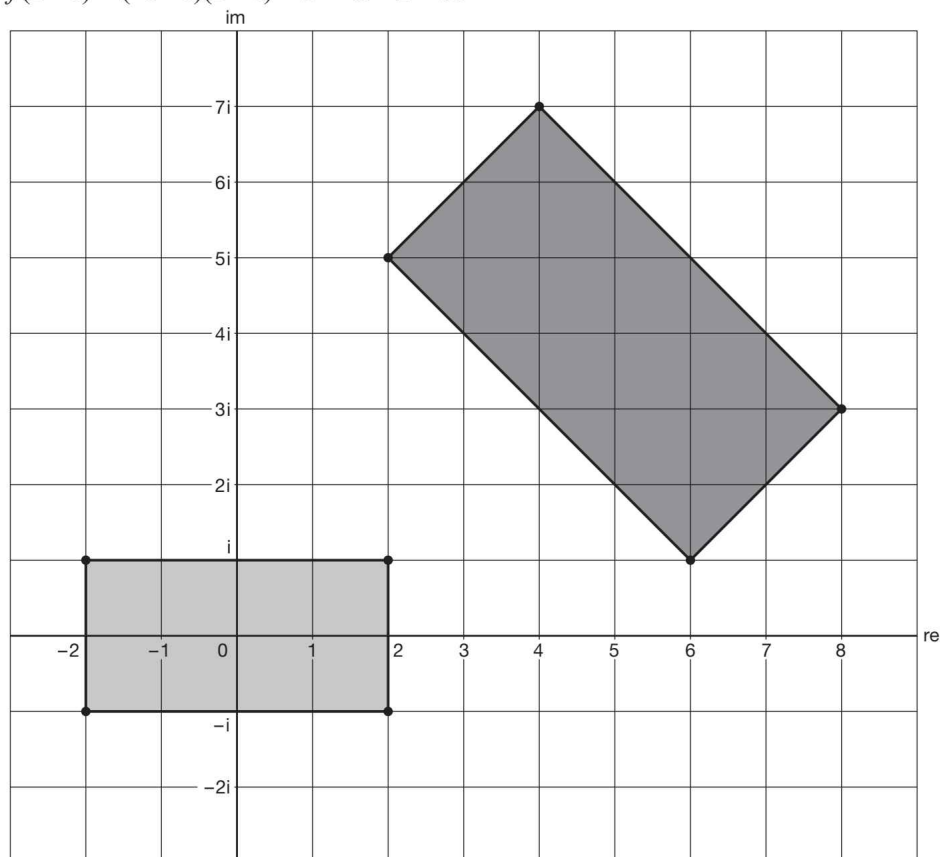
**Bladzijde 164**

- 15 a** $f(z) = 10 + i$ geeft $(-1 + i)z + 5 + 4i = 10 + i$
- $$(-1 + i)z = 5 - 3i$$
- $$z = \frac{5 - 3i}{-1 + i} \cdot \frac{-1 - i}{-1 - i} = \frac{-5 - 2i + 3i^2}{1 - i^2} = \frac{-8 - 2i}{2} = -4 - i$$

- b** $|-1 + i| = \sqrt{2}$ en $\text{Arg}(-1 + i) = \frac{3}{4}\pi$, dus rotatie om 0 over $\frac{3}{4}\pi$, vermenigvuldiging ten opzichte van 0 met $\sqrt{2}$ en translatie (5, 4).



- c** $f(-2 + i) = (-1 + i)(-2 + i) + 5 + 4i = 6 + i$
 $f(-2 - i) = (-1 + i)(-2 - i) + 5 + 4i = 8 + 3i$
 $f(2 - i) = (-1 + i)(2 - i) + 5 + 4i = 4 + 7i$
 $f(2 + i) = (-1 + i)(2 + i) + 5 + 4i = 2 + 5i$



- d** Het beeld van V heeft zijde $\sqrt{10}$.
 Dus V heeft zijde $\frac{\sqrt{10}}{\sqrt{2}} = \sqrt{5}$.
 De oppervlakte van V is dus 5.

16 a $f(z) = 0$ geeft $(1 + i\sqrt{3})z - 2 + i = 0$

$$z = \frac{2 - i}{1 + i\sqrt{3}} \cdot \frac{1 - i\sqrt{3}}{1 - i\sqrt{3}} = \frac{2 - 2i\sqrt{3} - i + i^2\sqrt{3}}{1 - 3i^2} = \frac{2 - \sqrt{3} - 2i\sqrt{3} - i}{4} = \frac{1}{2} - \frac{1}{4}\sqrt{3} + \left(-\frac{1}{4} - \frac{1}{2}\sqrt{3}\right)i$$

Het nulpunt van f is $\frac{1}{2} - \frac{1}{4}\sqrt{3} + \left(-\frac{1}{4} - \frac{1}{2}\sqrt{3}\right)i$.

$f(z) = z$ geeft $(1 + i\sqrt{3})z - 2 + i = z$

$$i\sqrt{3} \cdot z = 2 - i$$

$$z = \frac{2 - i}{i\sqrt{3}} \cdot \frac{i\sqrt{3}}{i\sqrt{3}} = \frac{2i\sqrt{3} - i^2\sqrt{3}}{3i^2} = \frac{2i\sqrt{3} + \sqrt{3}}{-3} = -\frac{1}{3}\sqrt{3} - \frac{2}{3}i\sqrt{3}$$

Het dekpunt van f is $-\frac{1}{3}\sqrt{3} - \frac{2}{3}i\sqrt{3}$.

b $g(z) = 0$ geeft $-2iz + 1 - 3i = 0$

$$-2iz = -1 + 3i$$

$$z = \frac{-1 + 3i}{-2i} \cdot \frac{i}{i} = \frac{-i + 3i^2}{-2i^2} = \frac{-i - 3}{2} = -1\frac{1}{2} - \frac{1}{2}i$$

Het nulpunt van g is $-1\frac{1}{2} - \frac{1}{2}i$.

$g(z) = z$ geeft $-2iz + 1 - 3i = z$

$$-z - 2iz = -1 + 3i$$

$$z + 2iz = 1 - 3i$$

$$z(1 + 2i) = 1 - 3i$$

$$z = \frac{1 - 3i}{1 + 2i} \cdot \frac{1 - 2i}{1 - 2i} = \frac{1 - 5i + 6i^2}{1 - 4i^2} = \frac{-5 - 5i}{5} = -1 - i$$

Het dekpunt van g is $-1 - i$.

17 a $f(z) = 0$ geeft $(\sqrt{3} + i)z + 1 + 2i = 0$

$$(\sqrt{3} + i)z = -1 - 2i$$

$$z = \frac{-1 - 2i}{\sqrt{3} + i} \cdot \frac{\sqrt{3} - i}{\sqrt{3} - i} = \frac{-\sqrt{3} + i - 2i\sqrt{3} + 2i^2}{3 - i^2} = \frac{-2 - \sqrt{3} + i - 2i\sqrt{3}}{4} = -\frac{1}{2} - \frac{1}{4}\sqrt{3} + \left(\frac{1}{4} - \frac{1}{2}\sqrt{3}\right)i$$

Het nulpunt van f is $-\frac{1}{2} - \frac{1}{4}\sqrt{3} + \left(\frac{1}{4} - \frac{1}{2}\sqrt{3}\right)i$.

$f(z) = z$ geeft $(\sqrt{3} + i)z + 1 + 2i = z$

$$(\sqrt{3} - 1 + i)z = -1 - 2i$$

$$z = \frac{-1 - 2i}{\sqrt{3} - 1 + i} \cdot \frac{\sqrt{3} - 1 - i}{\sqrt{3} - 1 - i} = \frac{-\sqrt{3} + 1 + i - 2i\sqrt{3} + 2i + 2i^2}{(\sqrt{3} - 1)^2 - i^2} = \frac{-\sqrt{3} + 1 + i - 2i\sqrt{3} + 2i - 2}{3 - 2\sqrt{3} + 1 + 1}$$

$$= \frac{-\sqrt{3} - 1 + 3i - 2i\sqrt{3}}{5 - 2\sqrt{3}} \cdot \frac{5 + 2\sqrt{3}}{5 + 2\sqrt{3}} = \frac{-5\sqrt{3} - 6 - 5 - 2\sqrt{3} + 15i + 6i\sqrt{3} - 10i\sqrt{3} - 12i}{25 - 12}$$

$$= \frac{-11 - 7\sqrt{3} + 3i - 4i\sqrt{3}}{13} = -\frac{11}{13} - \frac{7}{13}\sqrt{3} + \left(\frac{3}{13} - \frac{4}{13}\sqrt{3}\right)i$$

Het dekpunt van f is $-\frac{11}{13} - \frac{7}{13}\sqrt{3} + \left(\frac{3}{13} - \frac{4}{13}\sqrt{3}\right)i$.

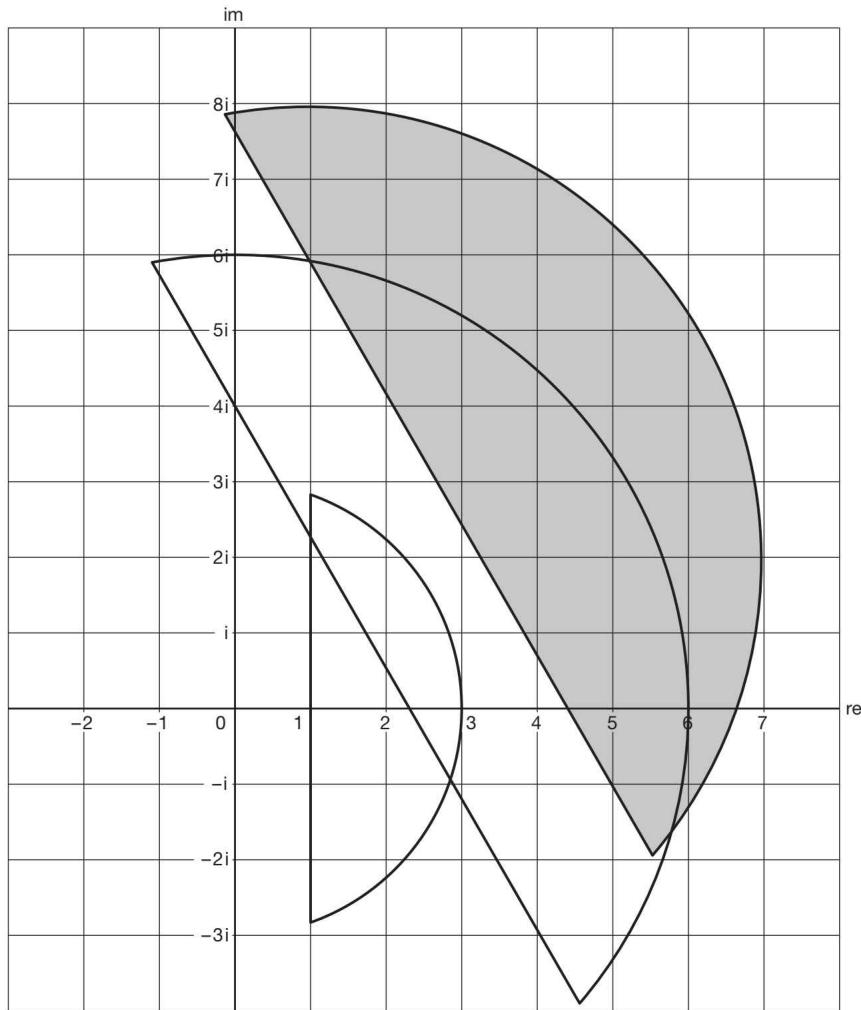
b $f(z) = 5 + 3i$ geeft $(\sqrt{3} + i)z + 1 + 2i = 5 + 3i$

$$(\sqrt{3} + i)z = 4 + i$$

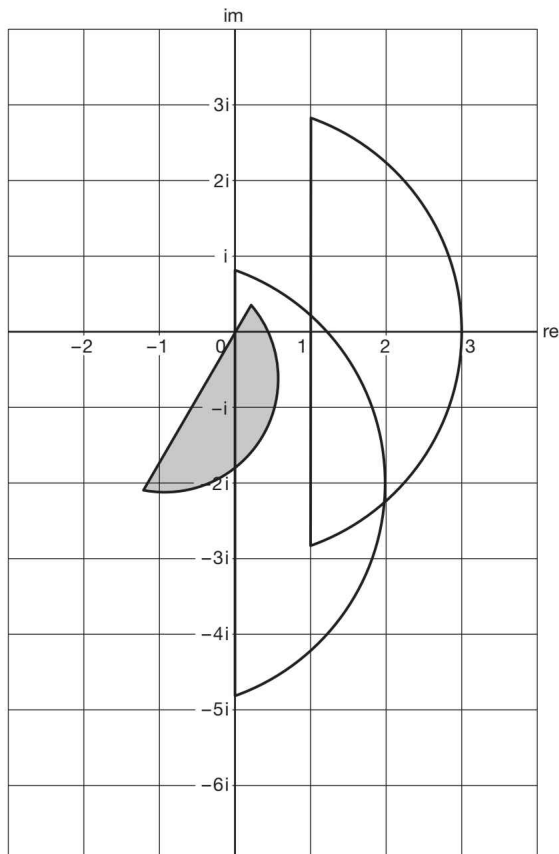
$$z = \frac{4 + i}{\sqrt{3} + i} \cdot \frac{\sqrt{3} - i}{\sqrt{3} - i} = \frac{4\sqrt{3} - 4i + i\sqrt{3} - i^2}{3 - i^2} = \frac{1 + 4\sqrt{3} - 4i + i\sqrt{3}}{4} = \frac{1}{4} + \sqrt{3} + \left(-1 + \frac{1}{4}\sqrt{3}\right)i$$

Het origineel van $5 + 3i$ is $\frac{1}{4} + \sqrt{3} + \left(-1 + \frac{1}{4}\sqrt{3}\right)i$.

- c $|\sqrt{3} + i| = 2$ en $\text{Arg}(\sqrt{3} + i) = \frac{1}{6}\pi$, dus rotatie om 0 over $\frac{1}{6}\pi$, vermenigvuldiging ten opzichte van 0 met 2 en translatie (1, 2).



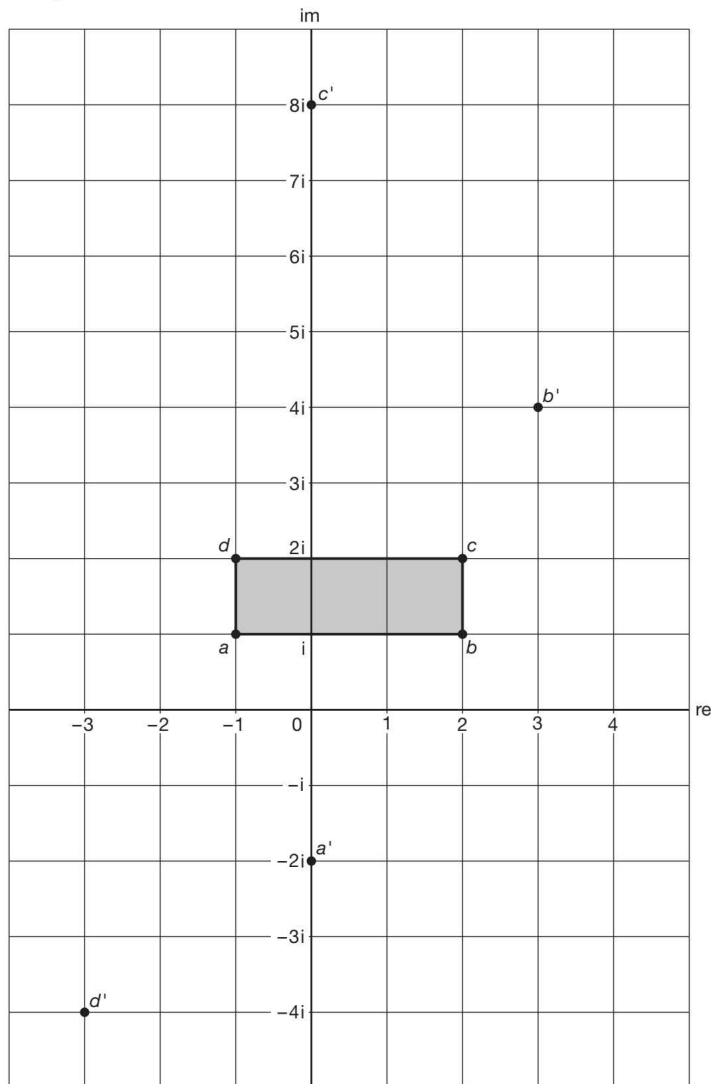
- d Translatie (-1, -2), vermenigvuldiging ten opzichte van 0 met $\frac{1}{2}$ en rotatie om 0 over $-\frac{1}{6}\pi$.



16.2 Niet-lineaire complexe functies

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b $a' = (-1 + i)^2 = 1 - 2i - 1 = -2i$

$b' = (2 + i)^2 = 4 + 4i - 1 = 3 + 4i$

$c' = (2 + 2i)^2 = 4 + 8i - 4 = 8i$

$d' = (-1 + 2i)^2 = 1 - 4i - 4 = -3 - 4i$

c Nee, want $-2i$, $3 + 4i$, $8i$ en $-3 - 4i$ zijn niet de hoekpunten van een rechthoek.

Bladzijde 168

19 a $f(z) = 0$ geeft $z^2 + 2 = 0$

$$z^2 = -2$$

$$z^2 = 2i^2$$

$$z = i\sqrt{2} \vee z = -i\sqrt{2}$$

De nulpunten zijn $i\sqrt{2}$ en $-i\sqrt{2}$.

b $f(z) = z$ geeft $z^2 + 2 = z$

$$z^2 - z + 2 = 0$$

$$\left(z - \frac{1}{2}\right)^2 - \frac{1}{4} + 2 = 0$$

$$\left(z - \frac{1}{2}\right)^2 = -1\frac{3}{4}$$

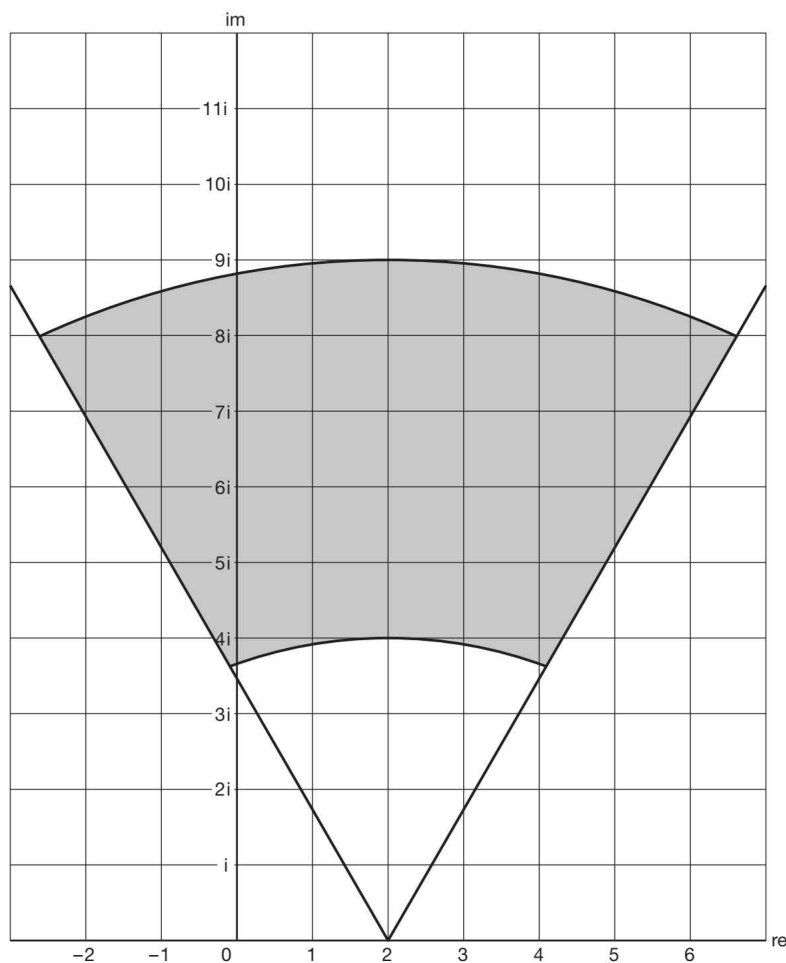
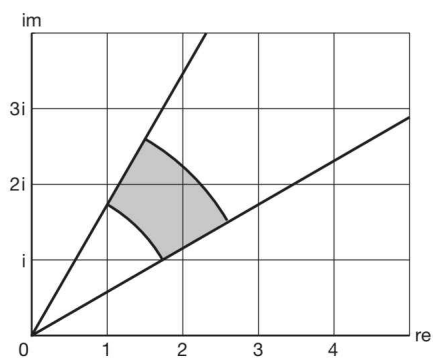
$$\left(z - \frac{1}{2}\right)^2 = \frac{7}{4}i^2$$

$$z - \frac{1}{2} = \frac{1}{2}i\sqrt{7} \vee z - \frac{1}{2} = -\frac{1}{2}i\sqrt{7}$$

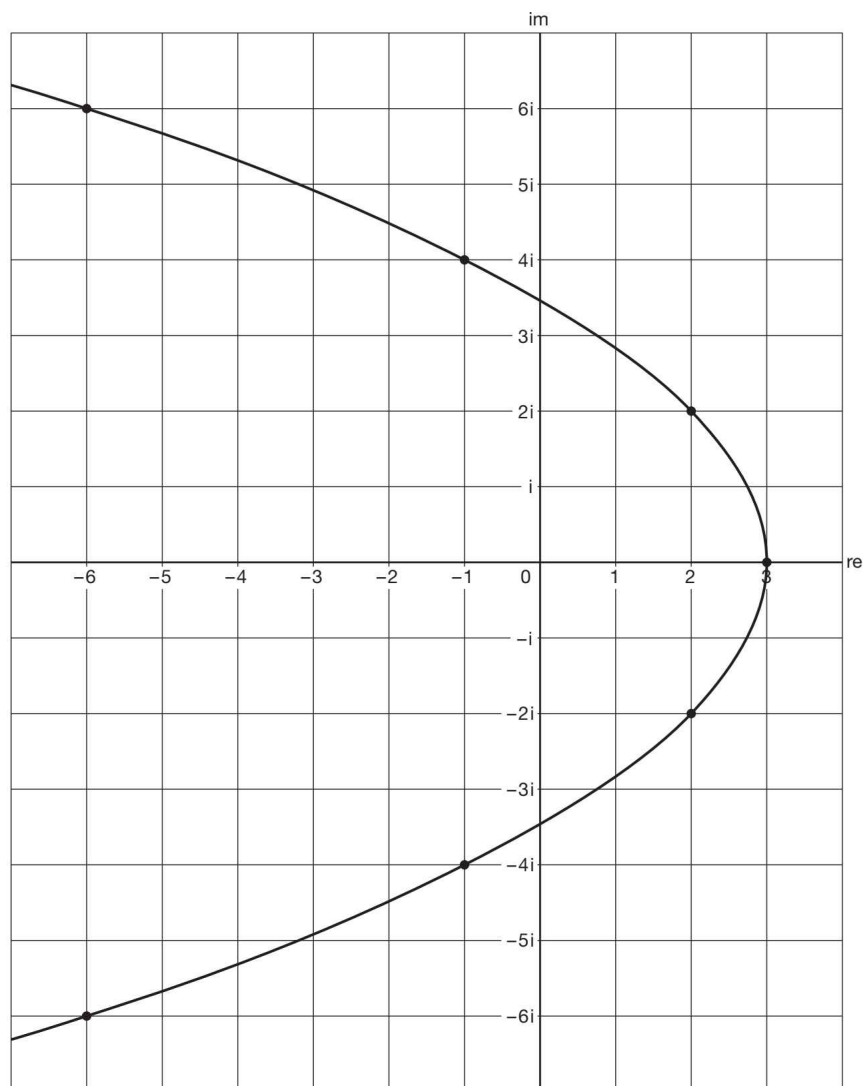
$$z = \frac{1}{2} + \frac{1}{2}i\sqrt{7} \vee z = \frac{1}{2} - \frac{1}{2}i\sqrt{7}$$

De dekpunten zijn $\frac{1}{2} + \frac{1}{2}i\sqrt{7}$ en $\frac{1}{2} - \frac{1}{2}i\sqrt{7}$.

- c Eerst z^2 : $|z^2| = |z|^2$, dus $4 \leq |z^2| \leq 9$ en $\arg(z^2) = 2 \cdot \arg(z)$, dus $\frac{1}{3}\pi \leq \arg(z^2) \leq \frac{2}{3}\pi$.
Dan de translatie $(2, 0)$.



- d $f(1) = 1^2 + 2 = 3$
 $f(1 + i) = (1 + i)^2 + 2 = 2 + 2i$
 $f(1 - i) = (1 - i)^2 + 2 = 2 - 2i$
 $f(1 + 2i) = (1 + 2i)^2 + 2 = -1 + 4i$
 $f(1 - 2i) = (1 - 2i)^2 + 2 = -1 - 4i$
 $f(1 + 3i) = (1 + 3i)^2 + 2 = -6 + 6i$
 $f(1 - 3i) = (1 - 3i)^2 + 2 = -6 - 6i$



20 a $f(z) = 0$ geeft $iz^2 - 4 = 0$

$$iz^2 = 4$$

$$z^2 = \frac{4}{i} = -4i$$

$$z^2 = 4e^{-\frac{1}{2}\pi i + k \cdot 2\pi i}$$

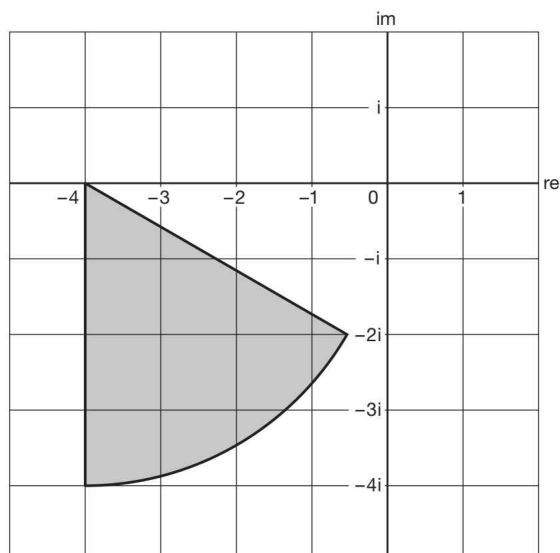
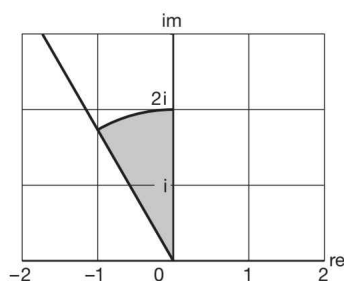
$$z = 2e^{-\frac{1}{4}\pi i + k \cdot \pi i}$$

$$z = \sqrt{2} - i\sqrt{2} \vee z = -\sqrt{2} + i\sqrt{2}$$

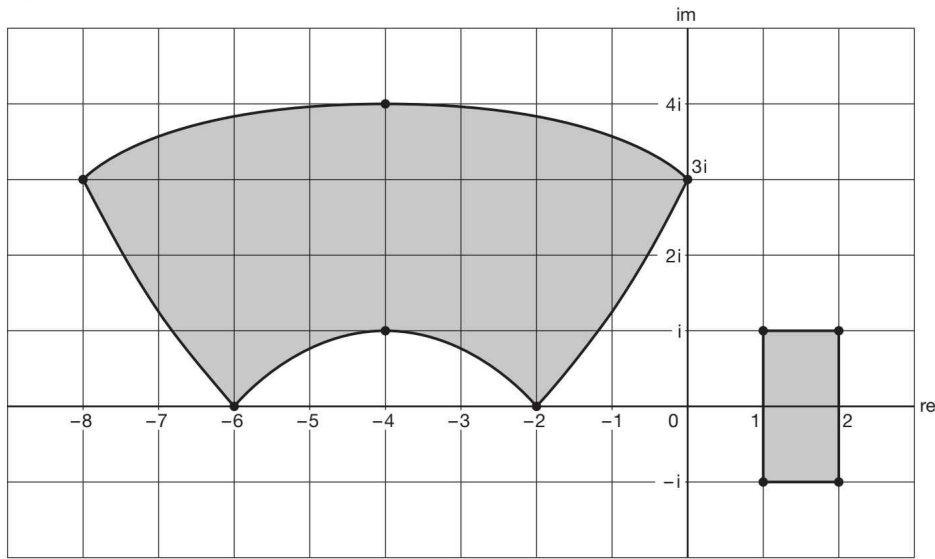
De nulpunten zijn $\sqrt{2} - i\sqrt{2}$ en $-\sqrt{2} + i\sqrt{2}$.

b Eerst iz^2 : $|iz^2| = |i| \cdot |z|^2 = 1 \cdot |z|^2$, dus $|iz^2| \leq 4$ en $\arg(iz^2) = \arg(i) + \arg(z^2) = \frac{1}{2}\pi + 2 \cdot \arg(z)$, dus $1\frac{1}{2}\pi \leq \arg(iz^2) \leq 1\frac{5}{6}\pi$.

Dan de translatie $(-4, 0)$.



$$\begin{aligned}
c \quad & f(1-i) = i(1-i)^2 - 4 = -2 \\
& f(2-i) = i(2-i)^2 - 4 = 3i \\
& f(2+i) = i(2+i)^2 - 4 = -8 + 3i \\
& f(1+i) = i(1+i)^2 - 4 = -6 \\
& f(2) = -4 + 4i \\
& f(1) = -4 + i
\end{aligned}$$

**Bladzijde 169**

- 21 a** Het beeld van $\operatorname{Re}(z) = 1$ en van $\operatorname{Re}(z) = -\operatorname{Im}(z)$ gaat door $-2 - 2i$.

Dus $z = 1 - i$.

$$f(1-i) = (1-i)^3 = (1-i)^2 \cdot (1-i) = -2i \cdot (1-i) = -2i - 2 = -2 - 2i \text{ klopt}$$

- b** $|z^3| = |z|^3$ en $\arg(z^3) = 3 \arg(z)$.

Voor alle punten van een lijn door 0 geldt dus dat ze op één lijn liggen waarvan de hoek ten opzichte van de reële as drie keer zo groot is geworden.

Het beeld van de lijn is dus weer een rechte lijn.

- c** Voor deze punten moet gelden $\arg(z) = \frac{1}{3}\pi \vee \arg(z) = -\frac{1}{3}\pi$.

$$\text{Dus } z = 1 + i\sqrt{3} \vee z = 1 - i\sqrt{3}.$$

- 22 a** $f(0) = e^0 = 1$

$$f(1) = e^1 = e$$

$$f(-2) = e^{-2} = \frac{1}{e^2}$$

- b** De functie $f(z) = e^z$ geeft voor elk reëel origineel een positief reëel beeld, dus de functie $f(z) = e^z$ beeldt de reële as af op de positieve reële as.

$$c \quad |f(\frac{1}{4}\pi i)| = |e^{\frac{1}{4}\pi i}| = |\cos(\frac{1}{4}\pi) + i \sin(\frac{1}{4}\pi)| = |\frac{1}{2}\sqrt{2} + \frac{1}{2}i\sqrt{2}| = \sqrt{(\frac{1}{2}\sqrt{2})^2 + (\frac{1}{2}\sqrt{2})^2} = \sqrt{\frac{1}{2} + \frac{1}{2}} = \sqrt{1} = 1$$

$$\operatorname{Arg}(f(\frac{1}{4}\pi i)) = \operatorname{Arg}(e^{\frac{1}{4}\pi i}) = \frac{1}{4}\pi$$

$$d \quad |f(\frac{3}{4}\pi i)| = |e^{\frac{3}{4}\pi i}| = |\cos(\frac{3}{4}\pi) + i \sin(\frac{3}{4}\pi)| = |-\frac{1}{2}\sqrt{2} + \frac{1}{2}i\sqrt{2}| = \sqrt{(-\frac{1}{2}\sqrt{2})^2 + (\frac{1}{2}\sqrt{2})^2} = \sqrt{\frac{1}{2} + \frac{1}{2}} = \sqrt{1} = 1$$

$$\operatorname{Arg}(f(\frac{3}{4}\pi i)) = \operatorname{Arg}(e^{\frac{3}{4}\pi i}) = \frac{3}{4}\pi$$

$$e \quad |f(\frac{5}{4}\pi i)| = |e^{\frac{5}{4}\pi i}| = |\cos(\frac{5}{4}\pi) + i \sin(\frac{5}{4}\pi)| = |-\frac{1}{2}\sqrt{2} - \frac{1}{2}i\sqrt{2}| = \sqrt{(-\frac{1}{2}\sqrt{2})^2 + (-\frac{1}{2}\sqrt{2})^2} = \sqrt{\frac{1}{2} + \frac{1}{2}} = \sqrt{1} = 1$$

$$\operatorname{Arg}(f(\frac{5}{4}\pi i)) = \operatorname{Arg}(e^{\frac{5}{4}\pi i}) = \frac{5}{4}\pi$$

$$f \quad |f(ai)| = |e^{ai}| = 1 \text{ en } \operatorname{Arg}(f(ai)) = \operatorname{Arg}(e^{ai}) = a$$

Dus het beeld van een zuiver imaginair getal ligt op de eenheidscirkel.

$$g \quad f(z + k \cdot 2\pi i) = e^{z + k \cdot 2\pi i} = e^z \cdot e^{k \cdot 2\pi i} = e^z \cdot (e^{2\pi i})^k = e^z \cdot 1^k = e^z$$

Dus de functie $f(z) = e^z$ is periodiek.

- 23 a** $\ln(e^{i\varphi}) = i\varphi$ met $\varphi = \pi$ geeft $\ln(e^{\pi i}) = \pi i$. En omdat $e^{\pi i} = -1$ geldt dus $\ln(-1) = \pi i$.

$$b \quad \ln(-1) = 3\pi i, \text{ want } e^{3\pi i} = e^{\pi i} = -1.$$

$$c \quad e^{\frac{1}{2}\pi i} = i, \text{ dus } f(i) = \ln(i) = \ln(e^{\frac{1}{2}\pi i}) = \frac{1}{2}\pi i.$$

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24 a $f(a + bi) = e^{a+bi} = e^a \cdot e^{bi} = e^a(\cos(b) + i\sin(b))$

Bij vaste a en variabele b krijg je dus de verzameling complexe getallen die op afstand e^a van 0 liggen en waarbij elke draaiingshoek b kan voorkomen.

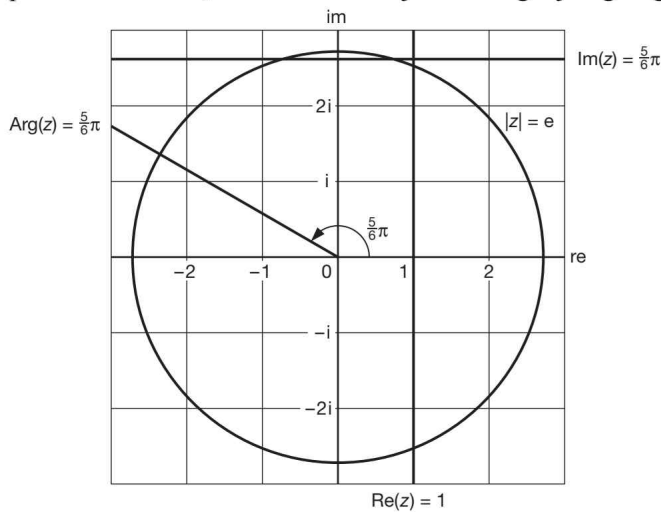
Dus bij de functie $f(z) = e^z$ is het beeld van $a + bi$ met a vast de cirkel met middelpunt 0 en straal e^a .

- b** Bij vaste a en vaste b is $e^a(\cos(b) + i\sin(b))$ een punt in het complexe vlak met afstand e^a tot 0 en draaiingshoek b om 0 ten opzichte van de positieve reële as.

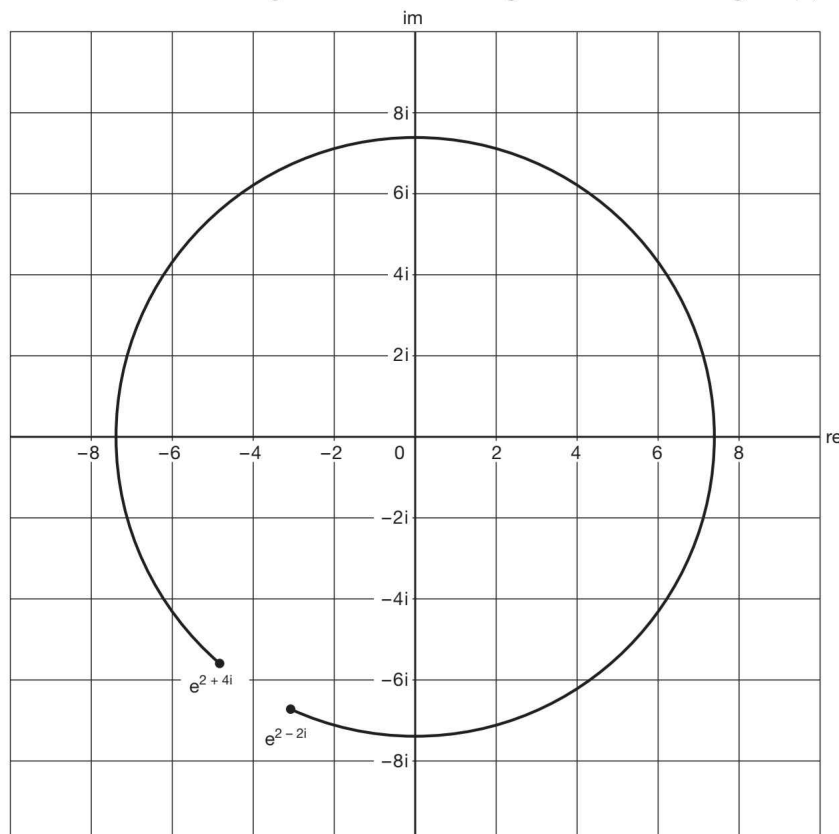
Bij vaste b en variabele a krijg je dus punten die alle hetzelfde argument hebben, maar waarvan de afstanden tot 0 verschillen. Deze punten vormen samen de halve lijn die een hoek van b radialen maakt met de positieve reële as.

Dus bij de functie $f(z) = e^z$ is het beeld van $a + bi$ met b vast de halve lijn die een hoek van b radialen maakt met de positieve reële as.

- 25** Het beeld van $\operatorname{Re}(z) = 1$ is de cirkel met middelpunt 0 en straal e^1 , ofwel de cirkel met vergelijking $|z| = e$. Het beeld van $\operatorname{Im}(z) = \frac{5}{6}\pi$ is de halve lijn met beginpunt 0 die een hoek van $\frac{5}{6}\pi$ radialen maakt met de positieve reële as, ofwel de halve lijn met vergelijking $\operatorname{Arg}(z) = \frac{5}{6}\pi$.



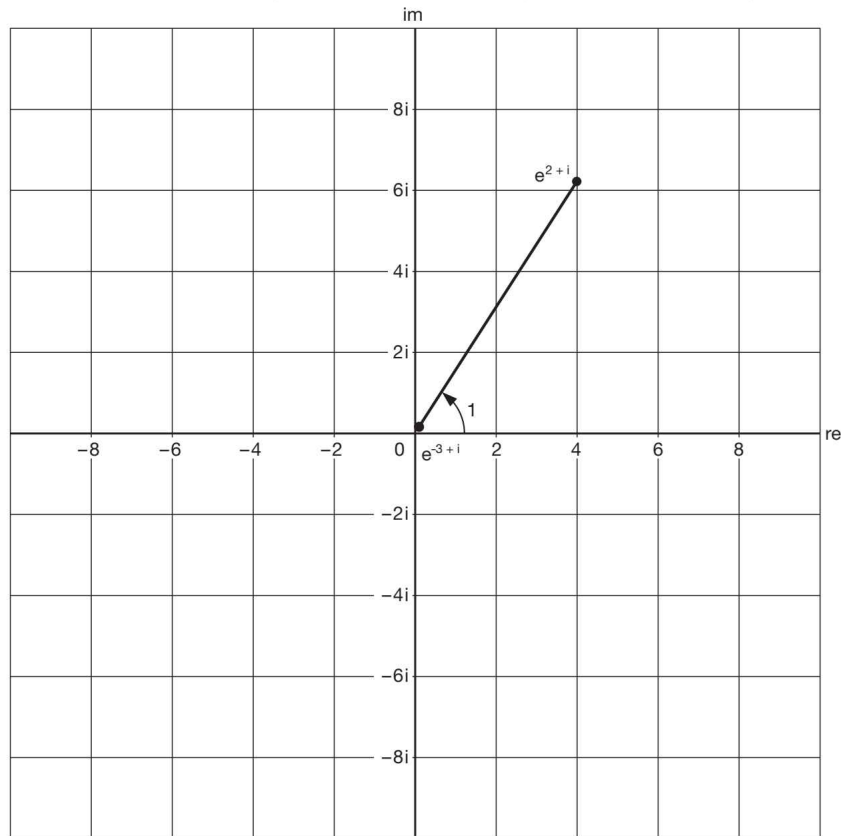
- 26 a** Het beeld van $\operatorname{Re}(z) = 2$ is de cirkel met middelpunt 0 en straal e^2 . Het beeld van $\operatorname{Im}(z) = 4$ heeft argument 4 radialen en het beeld van $\operatorname{Im}(z) = -2$ heeft argument -2 radialen. Dus het beeld van het lijnstuk bestaat uit de getallen z waarvoor geldt $|z| = e^2 \wedge -2 < \arg(z) < 4$.



b Het beeld van $\text{Im}(z) = 1$ bestaat uit getallen met argument 1 radiaal.

Het beeld van $\text{Re}(z) = -3$ heeft modulus e^{-3} en het beeld van $\text{Re}(z) = 2$ heeft modulus e^2 .

Dus het beeld van het lijnstuk bestaat uit alle getallen z waarvoor geldt $e^{-3} \leq |z| < e^2$ en $\text{Arg}(z) = 1$.



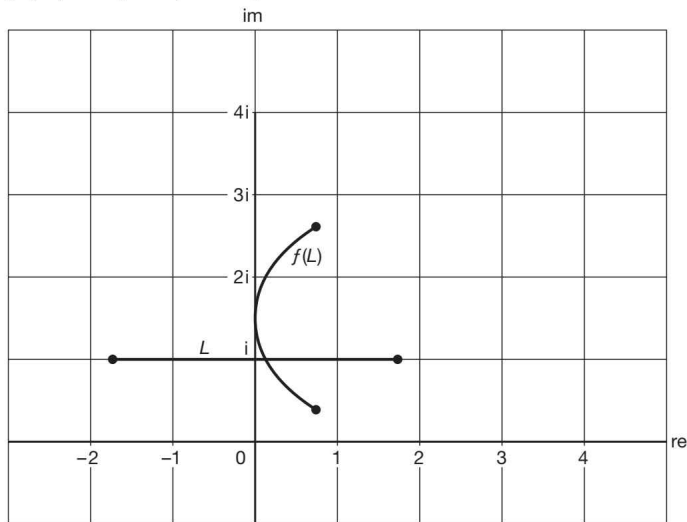
27 a $i^i = (e^{\frac{1}{2}\pi i})^i = e^{\frac{1}{2}\pi i^2} = e^{-\frac{1}{2}\pi}$
 b $i^{2i} = (e^{\frac{1}{2}\pi i})^{2i} = e^{\pi i^2} = e^{-\pi}$
 $i^1 + i = i^1 \cdot i^i = i \cdot e^{-\frac{1}{2}\pi}$

28 a $f(-e) = \ln(-e) = \ln(-1) + \ln(e) = \ln(e^{\pi i}) + 1 = 1 + \pi i$
 b $f(-e^2) = \ln(-e^2) = \ln(-1) + \ln(e^2) = \ln(e^{\pi i}) + 2 = 2 + \pi i$
 c $f(ei) = \ln(ei) = \ln(e) + \ln(i) = 1 + \ln(e^{\frac{1}{2}\pi i}) = \frac{1}{2}\pi i + 1$
 d $f(3i) = \ln(3i) = \ln(3) + \ln(i) = \ln(3) + \ln(e^{\frac{1}{2}\pi i}) = \frac{1}{2}\pi i + \ln(3)$
 e $f(-3) = \ln(-3) = \ln(-1) + \ln(3) = \ln(e^{\pi i}) + \ln(3) = \ln(3) + \pi i$
 f $f(-2i) = \ln(-2i) = \ln(2) + \ln(-i) = \ln(2) + \ln(e^{-\frac{1}{2}\pi i}) = \ln(2) - \frac{1}{2}\pi i$
 g $f(2-2i) = \ln(2-2i) = \ln(2\sqrt{2}e^{-\frac{1}{4}\pi i}) = \ln(2\sqrt{2}) + \ln(e^{-\frac{1}{4}\pi i}) = \frac{1}{2}\ln(2) - \frac{1}{4}\pi i$
 h $f(\sqrt{3}-i) = \ln(\sqrt{3}-i) = \ln(2e^{-\frac{1}{6}\pi i}) = \ln(2) + \ln(e^{-\frac{1}{6}\pi i}) = \ln(2) - \frac{1}{6}\pi i$
 i $f(e+ei) = \ln(e+ei) = \ln(e(1+i)) = \ln(e) + \ln(1+i) = 1 + \ln(\sqrt{2}e^{\frac{1}{4}\pi i}) = 1 + \ln(\sqrt{2}) + \ln(e^{\frac{1}{4}\pi i}) = 1 + \frac{1}{2}\ln(2) + \frac{1}{4}\pi i$

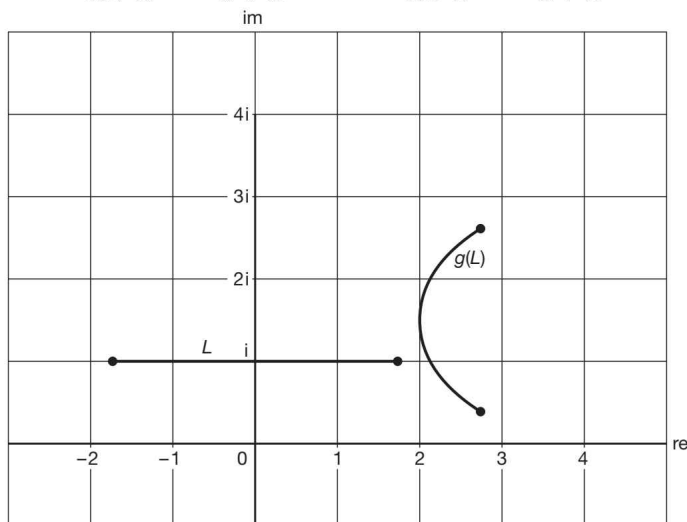
Bladzijde 172

29 a $f(-e\sqrt{e}) = \ln(-e\sqrt{e}) = \ln(-1) + \ln(e\sqrt{e}) = \ln(e^{\pi i}) + \ln(e^{\frac{1}{2}}) = \frac{1}{2} + \pi i$
 b $f(-i^2\sqrt{i}) = \ln(-i^2\sqrt{i}) = \ln(\sqrt{i}) = \frac{1}{2}\ln(i) = \frac{1}{2}\ln(e^{\frac{1}{2}\pi i}) = \frac{1}{2} \cdot \frac{1}{2}\pi i = \frac{1}{4}\pi i$
 c $f\left(-\frac{1}{e}\right) = \ln\left(-\frac{1}{e}\right) = \ln(-1) - \ln(e) = \ln(e^{\pi i}) - 1 = -1 + \pi i$
 d $f\left(\frac{2}{i\sqrt{i}}\right) = \ln\left(\frac{2}{i\sqrt{i}}\right) = \ln(2) - \ln(i\sqrt{i}) = \ln(2) - \ln(i^{\frac{1}{2}}) = \ln(2) - \frac{1}{2}\ln(i) = \ln(2) - \frac{1}{2}\ln(e^{\frac{1}{2}\pi i}) =$
 $\ln(2) - \frac{1}{2} \cdot \frac{1}{2}\pi i = \ln(2) - \frac{3}{4}\pi i$
 e $f\left(\frac{2}{1+i}\right) = \ln\left(\frac{2}{1+i}\right) = \ln(2) - \ln(1+i) = \ln(2) - \ln(\sqrt{2}e^{\frac{1}{4}\pi i}) = \ln(2) - \ln(\sqrt{2}) - \ln(e^{\frac{1}{4}\pi i}) =$
 $\ln(2) - \frac{1}{2}\ln(2) - \frac{1}{4}\pi i = \frac{1}{2}\ln(2) - \frac{1}{4}\pi i$
 f $f\left(\frac{e}{1+i\sqrt{3}}\right) = \ln\left(\frac{e}{1+i\sqrt{3}}\right) = \ln(e) - \ln(1+i\sqrt{3}) = 1 - \ln(2e^{\frac{1}{3}\pi i}) = 1 - \ln(2) - \ln(e^{\frac{1}{3}\pi i}) = 1 - \ln(2) - \frac{1}{3}\pi i$

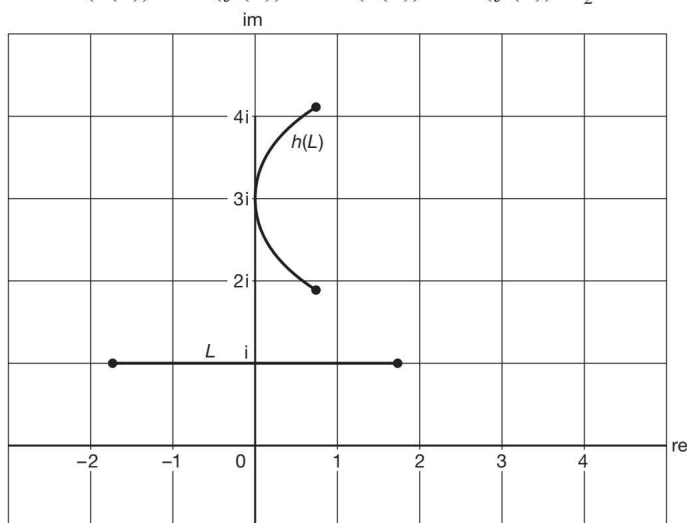
- 30 a** $f(\sqrt{3} + i) \approx 1,76 + 0,17i$
 $f(1 + i) \approx 0,35 + 0,79i$
 $f(1) \approx 1,57i$
 $f(-1 + i) \approx 0,35 + 2,36i$
 $f(-\sqrt{3} + i) \approx 0,69 + 2,62i$



- b** $g(e^2 z) = \ln(e^2 z) = \ln(e^2) + \ln(z) = 2 + \ln(z)$,
 dus $\operatorname{Re}(g(L)) = \operatorname{Re}(f(L)) + 2$ en $\operatorname{Im}(g(L)) = \operatorname{Im}(f(L))$.



- c** $h(iz) = \ln(iz) = \ln(i) + \ln(z) = \ln(e^{\frac{1}{2}\pi i}) + \ln(z) = \frac{1}{2}\pi i + \ln(z)$,
 dus $\operatorname{Re}(h(L)) = \operatorname{Re}(f(L))$ en $\operatorname{Im}(h(L)) = \operatorname{Im}(f(L)) + \frac{1}{2}\pi i$.



$$31 \quad \cos(i) = \frac{e^{i^2} + e^{-i^2}}{2} = \frac{e^{-1} + e^1}{2} = \frac{e^{-1}}{2} + \frac{e}{2} = \frac{1}{2e} + \frac{e}{2} = \frac{e}{2} + \frac{1}{2e}$$

Bladzijde 173

$$32 \quad a \quad f(i) = \sin(i) = \frac{e^{i^2} - e^{-i^2}}{2i} = \frac{e^{-1} - e^1}{2i} = \frac{1}{2ei} - \frac{e}{2i} = \left(\frac{e}{2} - \frac{1}{2e}\right)i$$

$$b \quad g\left(\frac{1}{6}\pi + i\right) = \cos\left(\frac{1}{6}\pi + i\right) = \frac{e^{i \cdot (\frac{1}{6}\pi + i)} + e^{-i \cdot (\frac{1}{6}\pi + i)}}{2} = \frac{e^{-1 + \frac{1}{6}\pi i} + e^{1 - \frac{1}{6}\pi i}}{2} = \frac{e^{-1} \cdot e^{\frac{1}{6}\pi i} + e^1 \cdot e^{-\frac{1}{6}\pi i}}{2} =$$

$$\frac{e^{-1}(\frac{1}{2}\sqrt{3} + \frac{1}{2}i) + e(\frac{1}{2}\sqrt{3} - \frac{1}{2}i)}{2} = \frac{\frac{1}{2}\sqrt{3}}{2e} + \frac{\frac{1}{2}i}{2e} + \frac{\frac{1}{2}e\sqrt{3}}{2} - \frac{\frac{1}{2}ei}{2} = \frac{\sqrt{3}}{4e} + \frac{e\sqrt{3}}{4} + \left(\frac{1}{4e} - \frac{e}{4}\right)i$$

$$c \quad f\left(\frac{1}{4}\pi - 3i\right) = \sin\left(\frac{1}{4}\pi - 3i\right) = \frac{e^{i \cdot (\frac{1}{4}\pi - 3i)} - e^{-i \cdot (\frac{1}{4}\pi - 3i)}}{2i} = \frac{e^{3 + \frac{1}{4}\pi i} - e^{-3 - \frac{1}{4}\pi i}}{2i} = \frac{e^3 \cdot e^{\frac{1}{4}\pi i} - e^{-3} \cdot e^{-\frac{1}{4}\pi i}}{2i} =$$

$$\frac{e^3(\frac{1}{2}\sqrt{2} + \frac{1}{2}i\sqrt{2}) - e^{-3}(\frac{1}{2}\sqrt{2} - \frac{1}{2}i\sqrt{2})}{2i} = \frac{\frac{1}{2}e^3\sqrt{2}}{2i} + \frac{\frac{1}{2}e^3i\sqrt{2}}{2i} - \frac{\frac{1}{2}e^{-3}\sqrt{2}}{2i} + \frac{\frac{1}{2}e^{-3}i\sqrt{2}}{2i} = \frac{-e^3i\sqrt{2}}{4} + \frac{e^3\sqrt{2}}{4} + \frac{i\sqrt{2}}{4e^3} + \frac{\sqrt{2}}{4e^3} =$$

$$\frac{e^3\sqrt{2}}{4} + \frac{\sqrt{2}}{4e^3} + \left(\frac{\sqrt{2}}{4e^3} - \frac{e^3\sqrt{2}}{4}\right)i$$

$$d \quad g\left(\frac{1}{3}\pi + 2i\right) = \cos\left(\frac{1}{3}\pi + 2i\right) = \frac{e^{i \cdot (\frac{1}{3}\pi + 2i)} + e^{-i \cdot (\frac{1}{3}\pi + 2i)}}{2} = \frac{e^{-2 + \frac{1}{3}\pi i} + e^{2 - \frac{1}{3}\pi i}}{2} = \frac{e^{-2} \cdot e^{\frac{1}{3}\pi i} + e^2 \cdot e^{-\frac{1}{3}\pi i}}{2} =$$

$$\frac{e^{-2}(\frac{1}{2} + \frac{1}{2}i\sqrt{3}) + e^2(\frac{1}{2} - \frac{1}{2}i\sqrt{3})}{2} = \frac{\frac{1}{2}}{2e^2} + \frac{\frac{1}{2}i\sqrt{3}}{2e^2} + \frac{\frac{1}{2}e^2}{2} - \frac{\frac{1}{2}e^2i\sqrt{3}}{2} = \frac{1}{4e^2} + \frac{e^2}{4} + \left(\frac{\sqrt{3}}{4e^2} - \frac{e^2\sqrt{3}}{4}\right)i$$

$$33 \quad a \quad \sin(2z) = \frac{e^{2iz} - e^{-2iz}}{2i}$$

$$2\sin(z)\cos(z) = 2 \cdot \frac{e^{iz} - e^{-iz}}{2i} \cdot \frac{e^{iz} + e^{-iz}}{2} = \frac{2((e^{iz})^2 - (e^{-iz})^2)}{4i} = \frac{e^{2iz} - e^{-2iz}}{2i}$$

Dus voor elk complex getal z geldt $\sin(2z) = 2\sin(z)\cos(z)$.

$$b \quad \sin^2(z) + \cos^2(z) = \left(\frac{e^{iz} - e^{-iz}}{2i}\right)^2 + \left(\frac{e^{iz} + e^{-iz}}{2}\right)^2 = \frac{e^{2iz} - 2e^0 + e^{-2iz}}{-4} + \frac{e^{2iz} + 2e^0 + e^{-2iz}}{4} =$$

$$\frac{-e^{2iz} + 2e^0 - e^{-2iz} + e^{2iz} + 2e^0 + e^{-2iz}}{4} = \frac{4}{4} = 1$$

Dus voor elk complex getal z geldt $\sin^2(z) + \cos^2(z) = 1$.

16.3 Meetkunde met complexe getallen**Bladzijde 175**

- 34 a $\vec{b} = \vec{a} + \vec{c}$, dus $b = a + c$.
 b $\vec{d} = \vec{c} + \vec{a}^*$ met $\vec{a}^*$ het beeld van $\vec{a}$ na draaiing over 90° , dus $d = c + ia$.
 c $e = a + c + ia$
 d $m = c + \frac{1}{2}a + \frac{1}{2}ia = \left(\frac{1}{2} + \frac{1}{2}i\right)a + c$

Bladzijde 176

$$35 \quad a \quad \vec{e} = \vec{a} + \vec{c} + \vec{a}_L$$

$$\vec{m} = \frac{1}{2}\vec{a} + \vec{c} + \frac{1}{2}\vec{a}_L$$

b Met complexe getallen kun je op dezelfde manier ook draaien over hoeken anders dan veelvoud van 90° .

- 36 a $c = a + (\cos(90^\circ) + i\sin(90^\circ))(b - a) = a + (0 + i)(b - a) = (1 - i)a + ib$
 b $c = a + (\cos(-90^\circ) + i\sin(-90^\circ))(b - a) = a + (0 - i)(b - a) = (1 + i)a - ib$
 c $c = a + (\cos(-60^\circ) + i\sin(-60^\circ))(b - a) = a + \left(\frac{1}{2} - \frac{1}{2}i\sqrt{3}\right)(b - a) = \left(\frac{1}{2} + \frac{1}{2}i\sqrt{3}\right)a + \left(\frac{1}{2} - \frac{1}{2}i\sqrt{3}\right)b$
 d $c = a + (\cos(30^\circ) + i\sin(30^\circ))(b - a) = a + \left(\frac{1}{2}\sqrt{3} + \frac{1}{2}i\right)(b - a) = \left(1 - \frac{1}{2}\sqrt{3} - \frac{1}{2}i\right)a + \left(\frac{1}{2}\sqrt{3} + \frac{1}{2}i\right)b$
 e $c = a + (\cos(45^\circ) + i\sin(45^\circ))(b - a) = a + \left(\frac{1}{2}\sqrt{2} + \frac{1}{2}i\sqrt{2}\right)(b - a) = \left(1 - \frac{1}{2}\sqrt{2} - \frac{1}{2}i\sqrt{2}\right)a + \left(\frac{1}{2}\sqrt{2} + \frac{1}{2}i\sqrt{2}\right)b$
 f $c = a + (\cos(135^\circ) + i\sin(135^\circ))(b - a) = a + \left(-\frac{1}{2}\sqrt{2} + \frac{1}{2}i\sqrt{2}\right)(b - a) = \left(1 + \frac{1}{2}\sqrt{2} - \frac{1}{2}i\sqrt{2}\right)a + \left(-\frac{1}{2}\sqrt{2} + \frac{1}{2}i\sqrt{2}\right)b$

Bladzijde 177

- 37 a**
- $(e - c)^*$
- is
- $e - c$
- gedraaid over
- 90°
- .

$$g = c - (e - c)^* = c - i(e - c) = c - ie + ic = (1 + i)c - ie$$

$$c = b + d \text{ en } e = d + (\cos(60^\circ) + i\sin(60^\circ))b = d + \left(\frac{1}{2} + \frac{1}{2}i\sqrt{3}\right)b$$

$$\text{Dit geeft } g = (1 + i)(b + d) - i\left(d + \left(\frac{1}{2} + \frac{1}{2}i\sqrt{3}\right)b\right)$$

$$= b + d + ib + id - id - \frac{1}{2}ib + \frac{1}{2}b\sqrt{3}$$

$$= \left(1 + \frac{1}{2}\sqrt{3} + \frac{1}{2}i\right)b + d$$

- b**
- $f = e + (c - e)i = ci + (1 - i)e$
-
- $c = b + d \text{ en } e = d + \left(\frac{1}{2} + \frac{1}{2}i\sqrt{3}\right)b$
- (zie onderdeel a)

$$\text{Dit geeft } f = (b + d)i + (1 - i)\left(d + \left(\frac{1}{2} + \frac{1}{2}i\sqrt{3}\right)b\right)$$

$$= ib + id + d + \frac{1}{2}b + \frac{1}{2}ib\sqrt{3} - id - \frac{1}{2}ib + \frac{1}{2}b\sqrt{3}$$

$$= \left(\frac{1}{2} + \frac{1}{2}\sqrt{3} + \frac{1}{2}i + \frac{1}{2}i\sqrt{3}\right)b + d$$

- 38 a**
- $c = b + (c - b) = b + (d - a) = b + d - a$

$$e = d - i(a - d) = -ia + (1 + i)d$$

$$f = a + i(d - a) = (1 - i)a + id$$

$$g = d + i(c - d) = d + i(b - a) = -ia + ib + d$$

$$m = a + \frac{1}{2}(d - a) + \frac{1}{2}i(d - a) = \left(\frac{1}{2} - \frac{1}{2}i\right)a + \left(\frac{1}{2} + \frac{1}{2}i\right)d$$

$$n = d + \frac{1}{2}(c - d) + \frac{1}{2}i(h - c)$$

$$= d + \frac{1}{2}(b - a) + \frac{1}{2}i(c - d)$$

$$= d + \frac{1}{2}(b - a) + \frac{1}{2}i(b - a)$$

$$= d + \frac{1}{2}b - \frac{1}{2}a + \frac{1}{2}ib - \frac{1}{2}ia$$

$$= \left(-\frac{1}{2} - \frac{1}{2}i\right)a + \left(\frac{1}{2} + \frac{1}{2}i\right)b + d$$

- b**
- $g = d + i(b - a)$
- , dus
- $g - a = d - a + i(b - a)$

- c**
- $e = d - i(a - d)$
- en
- $c = b + d - a$
- , dus

$$e - c = d - i(a - d) - (b + d - a) = d - i(a - d) - b - d + a = a - b - i(a - d) = (1 - i)a - b + id$$

$$i(g - a) = i(d - a + i(b - a)) = id - ia - (b - a) = (1 - i)a - b + id$$

$$\text{Dus } i(g - a) = e - c.$$

- d**
- De lijnstukken
- AG
- en
- CE
- zijn even lang en staan loodrecht op elkaar.

- 39**
- $c = a + (\cos(45^\circ) + i\sin(45^\circ))(b - a)$

$$= a + \left(\frac{1}{2}\sqrt{2} + \frac{1}{2}i\sqrt{2}\right)(b - a)$$

$$= \left(1 - \frac{1}{2}\sqrt{2} - \frac{1}{2}i\sqrt{2}\right)a + \left(\frac{1}{2}\sqrt{2} + \frac{1}{2}i\sqrt{2}\right)b$$

$$m = \frac{1}{2}(b + c)$$

$$= \frac{1}{2}\left(b + \left(1 - \frac{1}{2}\sqrt{2} - \frac{1}{2}i\sqrt{2}\right)a + \left(\frac{1}{2}\sqrt{2} + \frac{1}{2}i\sqrt{2}\right)b\right)$$

$$= \frac{1}{2}\left(\left(1 - \frac{1}{2}\sqrt{2} - \frac{1}{2}i\sqrt{2}\right)a + \left(1 + \frac{1}{2}\sqrt{2} + \frac{1}{2}i\sqrt{2}\right)b\right)$$

$$= \left(\frac{1}{2} - \frac{1}{4}\sqrt{2} - \frac{1}{4}i\sqrt{2}\right)a + \left(\frac{1}{2} + \frac{1}{4}\sqrt{2} + \frac{1}{4}i\sqrt{2}\right)b$$

$$\text{Dus } p = \frac{1}{2} - \frac{1}{4}\sqrt{2} - \frac{1}{4}i\sqrt{2} \text{ en } q = \frac{1}{2} + \frac{1}{4}\sqrt{2} + \frac{1}{4}i\sqrt{2}.$$

- 40**
- $d = a + (\cos(60^\circ) + i\sin(60^\circ))(b - a)$

$$= a + \left(\frac{1}{2} + \frac{1}{2}i\sqrt{3}\right)(b - a)$$

$$= \left(\frac{1}{2} - \frac{1}{2}i\sqrt{3}\right)a + \left(\frac{1}{2} + \frac{1}{2}i\sqrt{3}\right)b$$

$$c = b + (\cos(-90^\circ) + i\sin(-90^\circ))(d - b)$$

$$= b - id + ib$$

$$= (1 + i)b - id$$

$$= (1 + i)b - i\left(\left(\frac{1}{2} - \frac{1}{2}i\sqrt{3}\right)a + \left(\frac{1}{2} + \frac{1}{2}i\sqrt{3}\right)b\right)$$

$$= (1 + i)b + \left(-\frac{1}{2}i - \frac{1}{2}\sqrt{3}\right)a + \left(-\frac{1}{2}i + \frac{1}{2}\sqrt{3}\right)b$$

$$= \left(-\frac{1}{2}\sqrt{3} - \frac{1}{2}i\right)a + \left(1 + \frac{1}{2}\sqrt{3} + \frac{1}{2}i\right)b$$

$$\begin{aligned}
m &= \frac{1}{2}(c + d) \\
&= \frac{1}{2}\left(\left(-\frac{1}{2}\sqrt{3} - \frac{1}{2}i\right)a + \left(1 + \frac{1}{2}\sqrt{3} + \frac{1}{2}i\right)b + \left(\frac{1}{2} - \frac{1}{2}i\sqrt{3}\right)a + \left(\frac{1}{2} + \frac{1}{2}i\sqrt{3}\right)b\right) \\
&= \frac{1}{2}\left(\left(\frac{1}{2} - \frac{1}{2}\sqrt{3} - \frac{1}{2}i - \frac{1}{2}i\sqrt{3}\right)a + \left(1\frac{1}{2} + \frac{1}{2}\sqrt{3} + \frac{1}{2}i + \frac{1}{2}i\sqrt{3}\right)b\right) \\
&= \left(\frac{1}{4} - \frac{1}{4}\sqrt{3} - \frac{1}{4}i - \frac{1}{4}i\sqrt{3}\right)a + \left(\frac{3}{4} + \frac{1}{4}\sqrt{3} + \frac{1}{4}i + \frac{1}{4}i\sqrt{3}\right)b
\end{aligned}$$

Dus $p = -\frac{1}{4} - \frac{1}{4}\sqrt{3} - \frac{1}{4}i - \frac{1}{4}i\sqrt{3}$ en $q = \frac{3}{4} + \frac{1}{4}\sqrt{3} + \frac{1}{4}i + \frac{1}{4}i\sqrt{3}$.

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$$\begin{aligned}
41 \quad e &= a + (\cos(45^\circ) + i\sin(45^\circ))(c - a) \\
&= a + \left(\frac{1}{2}\sqrt{2} + \frac{1}{2}i\sqrt{2}\right)(c - a) \\
&= \left(1 - \frac{1}{2}\sqrt{2} - \frac{1}{2}i\sqrt{2}\right)a + \left(\frac{1}{2}\sqrt{2} + \frac{1}{2}i\sqrt{2}\right)c \\
d &= b + (\cos(-45^\circ) + i\sin(-45^\circ))(c - b) \\
&= b + \left(\frac{1}{2}\sqrt{2} - \frac{1}{2}i\sqrt{2}\right)(c - b) \\
&= \left(1 - \frac{1}{2}\sqrt{2} + \frac{1}{2}i\sqrt{2}\right)b + \left(\frac{1}{2}\sqrt{2} - \frac{1}{2}i\sqrt{2}\right)c
\end{aligned}$$

$$\begin{aligned}
m &= \frac{1}{2}(d + e) \\
&= \frac{1}{2}\left(\left(1 - \frac{1}{2}\sqrt{2} + \frac{1}{2}i\sqrt{2}\right)b + \left(\frac{1}{2}\sqrt{2} - \frac{1}{2}i\sqrt{2}\right)c + \left(1 - \frac{1}{2}\sqrt{2} - \frac{1}{2}i\sqrt{2}\right)a + \left(\frac{1}{2}\sqrt{2} + \frac{1}{2}i\sqrt{2}\right)c\right) \\
&= \frac{1}{2}\left(\left(1 - \frac{1}{2}\sqrt{2} - \frac{1}{2}i\sqrt{2}\right)a + \left(1 - \frac{1}{2}\sqrt{2} + \frac{1}{2}i\sqrt{2}\right)b + c\sqrt{2}\right) \\
&= \left(\frac{1}{2} - \frac{1}{4}\sqrt{2} - \frac{1}{4}i\sqrt{2}\right)a + \left(\frac{1}{2} - \frac{1}{4}\sqrt{2} + \frac{1}{4}i\sqrt{2}\right)b + \frac{1}{2}c\sqrt{2}
\end{aligned}$$

Dus $p = \frac{1}{2} - \frac{1}{4}\sqrt{2} - \frac{1}{4}i\sqrt{2}$, $q = \frac{1}{2} - \frac{1}{4}\sqrt{2} + \frac{1}{4}i\sqrt{2}$ en $r = \frac{1}{2}\sqrt{2}$.

$$\begin{aligned}
42 \quad a \quad q &= c + (\cos(60^\circ) + i\sin(60^\circ))(d - c) = c + \left(\frac{1}{2} + \frac{1}{2}i\sqrt{3}\right)(d - c) \\
r &= a + (\cos(60^\circ) + i\sin(60^\circ))(d - a) = a + \left(\frac{1}{2} + \frac{1}{2}i\sqrt{3}\right)(d - a) \\
q - r &= c + \left(\frac{1}{2} + \frac{1}{2}i\sqrt{3}\right)(d - c) - a - \left(\frac{1}{2} + \frac{1}{2}i\sqrt{3}\right)(d - a) \\
&= c + \frac{1}{2}d - \frac{1}{2}c + \frac{1}{2}id\sqrt{3} - \frac{1}{2}ic\sqrt{3} - a - \frac{1}{2}d + \frac{1}{2}a - \frac{1}{2}id\sqrt{3} + \frac{1}{2}ia\sqrt{3} \\
&= \frac{1}{2}c - \frac{1}{2}a + \left(\frac{1}{2}a\sqrt{3} - \frac{1}{2}c\sqrt{3}\right)i
\end{aligned}$$

$$\begin{aligned}
b \quad p &= c + (\cos(60^\circ) + i\sin(60^\circ))(b - c) = c + \left(\frac{1}{2} + \frac{1}{2}i\sqrt{3}\right)(b - c) \\
s &= a + (\cos(60^\circ) + i\sin(60^\circ))(b - a) = a + \left(\frac{1}{2} + \frac{1}{2}i\sqrt{3}\right)(b - a) \\
p - s &= c + \left(\frac{1}{2} + \frac{1}{2}i\sqrt{3}\right)(b - c) - a - \left(\frac{1}{2} + \frac{1}{2}i\sqrt{3}\right)(b - a) \\
&= c + \frac{1}{2}b - \frac{1}{2}c + \frac{1}{2}ib\sqrt{3} - \frac{1}{2}ic\sqrt{3} - a - \frac{1}{2}b + \frac{1}{2}a - \frac{1}{2}ib\sqrt{3} + \frac{1}{2}ia\sqrt{3} \\
&= \frac{1}{2}c - \frac{1}{2}a + \left(\frac{1}{2}a\sqrt{3} - \frac{1}{2}c\sqrt{3}\right)i
\end{aligned}$$

$$\begin{aligned}
c \quad p - q &= c + \left(\frac{1}{2} + \frac{1}{2}i\sqrt{3}\right)(b - c) - c - \left(\frac{1}{2} + \frac{1}{2}i\sqrt{3}\right)(d - c) \\
&= c + \frac{1}{2}b - \frac{1}{2}c + \frac{1}{2}ib\sqrt{3} - \frac{1}{2}ic\sqrt{3} - c - \frac{1}{2}d + \frac{1}{2}c - \frac{1}{2}id\sqrt{3} + \frac{1}{2}ic\sqrt{3} \\
&= \frac{1}{2}b - \frac{1}{2}d + \left(\frac{1}{2}b\sqrt{3} - \frac{1}{2}d\sqrt{3}\right)i \\
s - r &= a + \left(\frac{1}{2} + \frac{1}{2}i\sqrt{3}\right)(b - a) - a - \left(\frac{1}{2} + \frac{1}{2}i\sqrt{3}\right)(d - a) \\
&= a + \frac{1}{2}b - \frac{1}{2}a + \frac{1}{2}ib\sqrt{3} - \frac{1}{2}ia\sqrt{3} - a - \frac{1}{2}d + \frac{1}{2}a - \frac{1}{2}id\sqrt{3} + \frac{1}{2}ia\sqrt{3} \\
&= \frac{1}{2}b - \frac{1}{2}d + \left(\frac{1}{2}b\sqrt{3} - \frac{1}{2}d\sqrt{3}\right)i \\
p - q &= s - r, \text{ dus } PQ = RS \text{ en } PQ \parallel RS.
\end{aligned}$$

d Een parallelogram is een vierhoek met twee paar evenwijdige zijden.
 $PS \parallel QR$ en $PQ \parallel RS$, dus $PQRS$ is een parallelogram.

$$\begin{aligned}
43 \quad a \quad p &= c + (\cos(60^\circ) + i\sin(60^\circ))(b - c) = c + \left(\frac{1}{2} + \frac{1}{2}i\sqrt{3}\right)(b - c) = \frac{1}{2}b + \frac{1}{2}c + \frac{1}{2}ib\sqrt{3} - \frac{1}{2}ic\sqrt{3} \\
q &= a + (\cos(60^\circ) + i\sin(60^\circ))(c - a) = a + \left(\frac{1}{2} + \frac{1}{2}i\sqrt{3}\right)(c - a) = \frac{1}{2}a + \frac{1}{2}c - \frac{1}{2}ia\sqrt{3} + \frac{1}{2}ic\sqrt{3} \\
r &= b + (\cos(60^\circ) + i\sin(60^\circ))(a - b) = b + \left(\frac{1}{2} + \frac{1}{2}i\sqrt{3}\right)(a - b) = \frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}ia\sqrt{3} - \frac{1}{2}ib\sqrt{3} \\
d &= \frac{1}{3}(b + c + p) = \frac{1}{3}\left(b + c + \frac{1}{2}b + \frac{1}{2}c + \frac{1}{2}ib\sqrt{3} - \frac{1}{2}ic\sqrt{3}\right) = \frac{1}{3}\left(1\frac{1}{2}b + 1\frac{1}{2}c + \frac{1}{2}ib\sqrt{3} - \frac{1}{2}ic\sqrt{3}\right) \\
e &= \frac{1}{3}(a + c + q) = \frac{1}{3}\left(a + c + \frac{1}{2}a + \frac{1}{2}c - \frac{1}{2}ia\sqrt{3} + \frac{1}{2}ic\sqrt{3}\right) = \frac{1}{3}\left(1\frac{1}{2}a + 1\frac{1}{2}c - \frac{1}{2}ia\sqrt{3} + \frac{1}{2}ic\sqrt{3}\right) \\
f &= \frac{1}{3}(a + b + r) = \frac{1}{3}\left(a + b + \frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}ia\sqrt{3} - \frac{1}{2}ib\sqrt{3}\right) = \frac{1}{3}\left(1\frac{1}{2}a + 1\frac{1}{2}b + \frac{1}{2}ia\sqrt{3} - \frac{1}{2}ib\sqrt{3}\right)
\end{aligned}$$

$$\begin{aligned}
& d + (\cos(60^\circ) + i\sin(60^\circ))(e - d) = \\
& \frac{1}{3} \left(\frac{1}{2}b + \frac{1}{2}c + \frac{1}{2}ib\sqrt{3} - \frac{1}{2}ic\sqrt{3} \right) + \left(\frac{1}{2} + \frac{1}{2}i\sqrt{3} \right) \left(\frac{1}{3} \left(\frac{1}{2}a - \frac{1}{2}b - \frac{1}{2}ia\sqrt{3} - \frac{1}{2}ib\sqrt{3} + ic\sqrt{3} \right) \right) = \\
& \frac{1}{3} \left(\frac{1}{2}b + \frac{1}{2}c + \frac{1}{2}ib\sqrt{3} - \frac{1}{2}ic\sqrt{3} + \left(\frac{1}{2} + \frac{1}{2}i\sqrt{3} \right) \left(\frac{1}{2}a - \frac{1}{2}b - \frac{1}{2}ia\sqrt{3} - \frac{1}{2}ib\sqrt{3} + ic\sqrt{3} \right) \right) = \\
& \frac{1}{3} \left(\frac{1}{2}b + \frac{1}{2}c + \frac{1}{2}ib\sqrt{3} - \frac{1}{2}ic\sqrt{3} + \frac{3}{4}a - \frac{3}{4}b - \frac{1}{4}ia\sqrt{3} - \frac{1}{4}ib\sqrt{3} + \frac{1}{2}ic\sqrt{3} + \frac{3}{4}ia\sqrt{3} - \frac{3}{4}ib\sqrt{3} + \frac{3}{4}a + \frac{3}{4}b - \frac{1}{2}c \right) = \\
& \frac{1}{3} \left(\frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}ia\sqrt{3} - \frac{1}{2}ib\sqrt{3} \right) \\
& \text{Dus } f = d + (\cos(60^\circ) + i\sin(60^\circ))(e - d) \dots (1).
\end{aligned}$$

$$\begin{aligned}
& e + (\cos(60^\circ) + i\sin(60^\circ))(f - e) = \\
& \frac{1}{3} \left(\frac{1}{2}a + \frac{1}{2}c - \frac{1}{2}ia\sqrt{3} + \frac{1}{2}ic\sqrt{3} \right) + \left(\frac{1}{2} + \frac{1}{2}i\sqrt{3} \right) \left(\frac{1}{3} \left(\frac{1}{2}b - \frac{1}{2}c + ia\sqrt{3} - \frac{1}{2}ib\sqrt{3} - \frac{1}{2}ic\sqrt{3} \right) \right) = \\
& \frac{1}{3} \left(\frac{1}{2}a + \frac{1}{2}c - \frac{1}{2}ia\sqrt{3} + \frac{1}{2}ic\sqrt{3} + \left(\frac{1}{2} + \frac{1}{2}i\sqrt{3} \right) \left(\frac{1}{2}b - \frac{1}{2}c + ia\sqrt{3} - \frac{1}{2}ib\sqrt{3} - \frac{1}{2}ic\sqrt{3} \right) \right) = \\
& \frac{1}{3} \left(\frac{1}{2}a + \frac{1}{2}c - \frac{1}{2}ia\sqrt{3} + \frac{1}{2}ic\sqrt{3} + \frac{3}{4}b - \frac{3}{4}c + \frac{1}{2}ia\sqrt{3} - \frac{1}{4}ib\sqrt{3} - \frac{1}{4}ic\sqrt{3} + \frac{3}{4}ib\sqrt{3} - \frac{3}{4}ic\sqrt{3} - \frac{1}{2}a + \frac{3}{4}b + \frac{3}{4}c \right) = \\
& \frac{1}{3} \left(\frac{1}{2}b + \frac{1}{2}c + \frac{1}{2}ib\sqrt{3} - \frac{1}{2}ic\sqrt{3} \right) \\
& \text{Dus } d = e + (\cos(60^\circ) + i\sin(60^\circ))(f - e) \dots (2).
\end{aligned}$$

Uit (1) en (2) volgt dat driehoek DEF een gelijkzijdige driehoek is.

Hiermee is de stelling bewezen.

b Als de drie gelijkzijdige driehoeken naar binnen zijn gericht geldt

$$\begin{aligned}
p &= c + (\cos(-60^\circ) + i\sin(-60^\circ))(b - c) = c + \left(\frac{1}{2} - \frac{1}{2}i\sqrt{3} \right) (b - c) = \frac{1}{2}b + \frac{1}{2}c - \frac{1}{2}ib\sqrt{3} + \frac{1}{2}ic\sqrt{3} \\
q &= a + (\cos(-60^\circ) + i\sin(-60^\circ))(c - a) = a + \left(\frac{1}{2} - \frac{1}{2}i\sqrt{3} \right) (c - a) = \frac{1}{2}a + \frac{1}{2}c + \frac{1}{2}ia\sqrt{3} - \frac{1}{2}ic\sqrt{3} \\
r &= b + (\cos(-60^\circ) + i\sin(-60^\circ))(a - b) = b + \left(\frac{1}{2} - \frac{1}{2}i\sqrt{3} \right) (a - b) = \frac{1}{2}a + \frac{1}{2}b - \frac{1}{2}ia\sqrt{3} + \frac{1}{2}ib\sqrt{3} \\
d &= \frac{1}{3}(b + c + p) = \frac{1}{3} \left(b + c + \frac{1}{2}b + \frac{1}{2}c - \frac{1}{2}ib\sqrt{3} + \frac{1}{2}ic\sqrt{3} \right) = \frac{1}{3} \left(\frac{1}{2}b + \frac{1}{2}c - \frac{1}{2}ib\sqrt{3} + \frac{1}{2}ic\sqrt{3} \right) \\
e &= \frac{1}{3}(a + c + q) = \frac{1}{3} \left(a + c + \frac{1}{2}a + \frac{1}{2}c + \frac{1}{2}ia\sqrt{3} - \frac{1}{2}ic\sqrt{3} \right) = \frac{1}{3} \left(\frac{1}{2}a + \frac{1}{2}c + \frac{1}{2}ia\sqrt{3} - \frac{1}{2}ic\sqrt{3} \right) \\
f &= \frac{1}{3}(a + b + r) = \frac{1}{3} \left(a + b + \frac{1}{2}a + \frac{1}{2}b - \frac{1}{2}ia\sqrt{3} + \frac{1}{2}ib\sqrt{3} \right) = \frac{1}{3} \left(\frac{1}{2}a + \frac{1}{2}b - \frac{1}{2}ia\sqrt{3} + \frac{1}{2}ib\sqrt{3} \right) \\
& d + (\cos(-60^\circ) + i\sin(-60^\circ))(e - d) = \\
& \frac{1}{3} \left(\frac{1}{2}b + \frac{1}{2}c - \frac{1}{2}ib\sqrt{3} + \frac{1}{2}ic\sqrt{3} \right) + \left(\frac{1}{2} - \frac{1}{2}i\sqrt{3} \right) \left(\frac{1}{3} \left(\frac{1}{2}a - \frac{1}{2}b + \frac{1}{2}ia\sqrt{3} + \frac{1}{2}ib\sqrt{3} - ic\sqrt{3} \right) \right) = \\
& \frac{1}{3} \left(\frac{1}{2}b + \frac{1}{2}c - \frac{1}{2}ib\sqrt{3} + \frac{1}{2}ic\sqrt{3} + \left(\frac{1}{2} - \frac{1}{2}i\sqrt{3} \right) \left(\frac{1}{2}a - \frac{1}{2}b + \frac{1}{2}ia\sqrt{3} + \frac{1}{2}ib\sqrt{3} - ic\sqrt{3} \right) \right) = \\
& \frac{1}{3} \left(\frac{1}{2}b + \frac{1}{2}c - \frac{1}{2}ib\sqrt{3} + \frac{1}{2}ic\sqrt{3} + \frac{3}{4}a - \frac{3}{4}b + \frac{1}{4}ia\sqrt{3} + \frac{1}{4}ib\sqrt{3} - \frac{1}{2}ic\sqrt{3} - \frac{3}{4}ia\sqrt{3} + \frac{3}{4}ib\sqrt{3} + \frac{3}{4}a + \frac{3}{4}b - \frac{1}{2}c \right) = \\
& \frac{1}{3} \left(\frac{1}{2}a + \frac{1}{2}b - \frac{1}{2}ia\sqrt{3} + \frac{1}{2}ib\sqrt{3} \right) \\
& \text{Dus } f = d + (\cos(-60^\circ) + i\sin(-60^\circ))(e - d) \dots (1).
\end{aligned}$$

$$\begin{aligned}
& e + (\cos(-60^\circ) + i\sin(-60^\circ))(f - e) = \\
& \frac{1}{3} \left(\frac{1}{2}a + \frac{1}{2}c + \frac{1}{2}ia\sqrt{3} - \frac{1}{2}ic\sqrt{3} \right) + \left(\frac{1}{2} - \frac{1}{2}i\sqrt{3} \right) \left(\frac{1}{3} \left(\frac{1}{2}b - \frac{1}{2}c - ia\sqrt{3} + \frac{1}{2}ib\sqrt{3} + \frac{1}{2}ic\sqrt{3} \right) \right) = \\
& \frac{1}{3} \left(\frac{1}{2}a + \frac{1}{2}c + \frac{1}{2}ia\sqrt{3} - \frac{1}{2}ic\sqrt{3} + \left(\frac{1}{2} - \frac{1}{2}i\sqrt{3} \right) \left(\frac{1}{2}b - \frac{1}{2}c - ia\sqrt{3} + \frac{1}{2}ib\sqrt{3} + \frac{1}{2}ic\sqrt{3} \right) \right) = \\
& \frac{1}{3} \left(\frac{1}{2}a + \frac{1}{2}c + \frac{1}{2}ia\sqrt{3} - \frac{1}{2}ic\sqrt{3} + \frac{3}{4}b - \frac{3}{4}c - \frac{1}{2}ia\sqrt{3} + \frac{1}{4}ib\sqrt{3} + \frac{1}{4}ic\sqrt{3} - \frac{3}{4}ib\sqrt{3} + \frac{3}{4}ic\sqrt{3} - \frac{1}{2}a + \frac{3}{4}b + \frac{3}{4}c \right) = \\
& \frac{1}{3} \left(\frac{1}{2}b + \frac{1}{2}c - \frac{1}{2}ib\sqrt{3} + \frac{1}{2}ic\sqrt{3} \right) \\
& \text{Dus } d = e + (\cos(-60^\circ) + i\sin(-60^\circ))(f - e) \dots (2).
\end{aligned}$$

Uit (1) en (2) volgt dat driehoek DEF een gelijkzijdige driehoek is.

Hiermee is bewezen dat de stelling ook geldig is als de drie gelijkzijdige driehoeken naar binnen zijn gericht.

16.4 Derdegraadsvergelijkingen

Bladzijde 180

44 a $x = 1$ geeft $1^3 + 1^2 + 1 - 3 = 1 + 1 + 1 - 3 = 0$ klopt.

Dus $x = 1$ is een oplossing van de vergelijking $x^3 + x^2 + x - 3 = 0$.

b $x = -1 + i\sqrt{2}$ geeft $(-1 + i\sqrt{2})^3 + (-1 + i\sqrt{2})^2 + (-1 + i\sqrt{2}) - 3 =$
 $(-1 + i\sqrt{2})((-1 + i\sqrt{2})^2 + (-1 + i\sqrt{2}) + 1) - 3 =$
 $(-1 + i\sqrt{2})(1 - 2i\sqrt{2} - 2 - 1 + i\sqrt{2} + 1) - 3 =$
 $(-1 + i\sqrt{2})(-1 - i\sqrt{2}) - 3 =$
 $1 + 2 - 3 = 0$ klopt.

Dus $x = -1 + i\sqrt{2}$ is een oplossing van de vergelijking $x^3 + x^2 + x - 3 = 0$.

Bladzijde 181

- 45 a**
- Voer in
- $y_1 = x^3 - 4x^2 + 4x - 16$
- .

De optie zero (TI) of ROOT (Casio) geeft $x = 4$.

$$z - 4 \mid z^3 - 4z^2 + 4z - 16 \setminus z^2 + 4$$

$$\begin{array}{r} z^3 - 4z^2 \\ \hline 4z - 16 \\ 4z - 16 \\ \hline 0 \end{array}$$

$$z^2 + 4 = 0$$

$$z^2 = -4$$

$$z^2 = 4i^2$$

$$z = 2i \vee z = -2i$$

$$\text{Dus } z = 4 \vee z = 2i \vee z = -2i.$$

- b**
- Voer in
- $y_1 = x^3 + 2x - 12$
- .

De optie zero (TI) of ROOT (Casio) geeft $x = 2$.

$$z - 2 \mid z^3 + 2z - 12 \setminus z^2 + 2z + 6$$

$$\begin{array}{r} z^3 - 2z^2 \\ \hline 2z^2 + 2z \\ 2z^2 - 4z \\ \hline 6z - 12 \\ 6z - 12 \\ \hline 0 \end{array}$$

$$z^2 + 2z + 6 = 0$$

$$(z + 1)^2 - 1 + 6 = 0$$

$$(z + 1)^2 = -5$$

$$(z + 1)^2 = 5i^2$$

$$z + 1 = i\sqrt{5} \vee z + 1 = -i\sqrt{5}$$

$$z = -1 + i\sqrt{5} \vee z = -1 - i\sqrt{5}$$

$$\text{Dus } z = 2 \vee z = -1 + i\sqrt{5} \vee z = -1 - i\sqrt{5}.$$

- c**
- $2z^3 + 3z^2 - 8z = 30$
- ofwel
- $z^3 + \frac{1}{2}z^2 - 4z - 15 = 0$

Voer in $y_1 = x^3 + \frac{1}{2}x^2 - 4x - 15$.De optie zero (TI) of ROOT (Casio) geeft $x = 2\frac{1}{2}$.

$$z - 2\frac{1}{2} \mid z^3 + \frac{1}{2}z^2 - 4z - 15 \setminus z^2 + 4z + 6$$

$$\begin{array}{r} z^3 - 2\frac{1}{2}z^2 \\ \hline 4z^2 - 4z \\ 4z^2 - 10z \\ \hline 6z - 15 \\ 6z - 15 \\ \hline 0 \end{array}$$

$$z^2 + 4z + 6 = 0$$

$$(z + 2)^2 - 4 + 6 = 0$$

$$(z + 2)^2 = -2$$

$$(z + 2)^2 = 2i^2$$

$$z + 2 = i\sqrt{2} \vee z + 2 = -i\sqrt{2}$$

$$z = -2 + i\sqrt{2} \vee z = -2 - i\sqrt{2}$$

$$\text{Dus } z = 2\frac{1}{2} \vee z = -2 + i\sqrt{2} \vee z = -2 - i\sqrt{2}.$$

- d**
- $2z^3 - z^2 + 3z + 2 = 0$
- ofwel
- $z^3 - \frac{1}{2}z^2 + \frac{3}{2}z + 1 = 0$

Voer in $y_1 = x^3 - \frac{1}{2}x^2 + \frac{3}{2}x + 1$.De optie zero (TI) of ROOT (Casio) geeft $x = -\frac{1}{2}$.

$$z + \frac{1}{2} \mid z^3 - \frac{1}{2}z^2 + \frac{3}{2}z + 1 \setminus z^2 - z + 2$$

$$\begin{array}{r} z^3 + \frac{1}{2}z^2 \\ \hline -z^2 + 1\frac{1}{2}z \\ -z^2 - \frac{1}{2}z \\ \hline 2z + 1 \\ 2z + 1 \\ \hline 0 \end{array}$$

$$z^2 - z + 2 = 0$$

$$\left(z - \frac{1}{2}\right)^2 - \frac{1}{4} + 2 = 0$$

$$\left(z - \frac{1}{2}\right)^2 = -1\frac{3}{4}$$

$$\left(z - \frac{1}{2}\right)^2 = \frac{7}{4}i^2$$

$$z - \frac{1}{2} = \frac{1}{2}i\sqrt{7} \vee z - \frac{1}{2} = -\frac{1}{2}i\sqrt{7}$$

$$z = \frac{1}{2} + \frac{1}{2}i\sqrt{7} \vee z = \frac{1}{2} - \frac{1}{2}i\sqrt{7}$$

$$\text{Dus } z = -\frac{1}{2} \vee z = \frac{1}{2} + \frac{1}{2}i\sqrt{7} \vee z = \frac{1}{2} - \frac{1}{2}i\sqrt{7}.$$

46 a $(u+v)^3 + (-3uv)(u+v) = q$
 $u^3 + 3u^2v + 3uv^2 + v^3 - 3u^2v - 3uv^2 = q$
 $u^3 + v^3 = q$

b $v = -\frac{p}{3u}$ geeft $u^3 + \left(-\frac{p}{3u}\right)^3 = q$

$$u^3 - \frac{p^3}{27u^3} = q$$

$$u^6 - \frac{1}{27}p^3 = qu^3$$

$$u^6 - qu^3 - \frac{1}{27}p^3 = 0$$

Bladzijde 182

47 a $(u+v)^3 + 6(u+v) = 20$
 $u^3 + 3u^2v + 3uv^2 + v^3 + 6u + 6v = 20$
 $u^3 + 3u \cdot uv + 3v \cdot uv + v^3 + 6u + 6v = 20$
 $u^3 + 3u \cdot -2 + 3v \cdot -2 + v^3 + 6u + 6v = 20$
 $u^3 - 6u - 6v + v^3 + 6u + 6v = 20$
 $u^3 + v^3 = 20$

b $v = -\frac{2}{u}$ geeft $u^3 + \left(-\frac{2}{u}\right)^3 = 20$

$$u^3 - \frac{8}{u^3} = 20$$

$$u^6 - 8 = 20u^3$$

$$u^6 - 20u^3 - 8 = 0$$

c Stel $x = u^3$.

$$x^2 - 20x - 8 = 0$$

$$D = (-20)^2 - 4 \cdot 1 \cdot -8 = 432$$

$$x = \frac{20 - \sqrt{432}}{2} \vee x = \frac{20 + \sqrt{432}}{2}$$

$$u^3 = \frac{20 - \sqrt{432}}{2} \vee u^3 = \frac{20 + \sqrt{432}}{2}$$

d $u^3 = 10 + 6\sqrt{3}$ geeft $10 + 6\sqrt{3} + v^3 = 20$

$$v^3 = 20 - 10 - 6\sqrt{3}$$

$$v^3 = 10 - 6\sqrt{3}$$

e $u^3 = 10 + 6\sqrt{3}$ geeft $u = \sqrt[3]{10 + 6\sqrt{3}}$

$$v^3 = 10 - 6\sqrt{3}$$
 geeft $v = \sqrt[3]{10 - 6\sqrt{3}}$

$$z = u + v = \sqrt[3]{10 + 6\sqrt{3}} + \sqrt[3]{10 - 6\sqrt{3}}$$

f De GR geeft $\sqrt[3]{10 + 6\sqrt{3}} + \sqrt[3]{10 - 6\sqrt{3}} = 2$.

$$z = 2 \text{ geeft } 2^3 + 6 \cdot 2 = 8 + 12 = 20 \text{ klopt.}$$

$$\text{Dus } z = 2 \text{ is een oplossing van de vergelijking } z^3 + 6z = 20.$$

Bladzijde 183

- 48** Verder rekenen met $u^3 = 10 - 6\sqrt{3}$ geeft $v^3 = 20 - u^3 = 20 - (10 - 6\sqrt{3}) = 10 + 6\sqrt{3}$.

Dit geeft $z = u + v = \sqrt[3]{10 - 6\sqrt{3}} + \sqrt[3]{10 + 6\sqrt{3}}$ en dus hetzelfde resultaat.

Verder rekenen met $u^3 = -1$ geeft $v^3 = 63 - u^3 = 63 - (-1) = 64$.

Dit geeft $z = u + v = \sqrt[3]{-1} + \sqrt[3]{64} = -1 + 4 = 3$ en dus hetzelfde resultaat.

- 49 a** $z = u + v$ en $36 = -3uv$ geeft $u^3 + v^3 = 208$ $\left\{ \begin{array}{l} u^3 + \left(-\frac{12}{u}\right)^3 = 208 \\ u^3 - \frac{1728}{u^3} = 208 \end{array} \right.$

Uit $-3uv = 36$ volgt $v = -\frac{12}{u}$.

$$u^3 - \frac{1728}{u^3} = 208$$

$$u^6 - 208u^3 - 1728 = 0$$

$$D = 50176, \text{ dus } \sqrt{D} = 224$$

$$u^3 = \frac{208 + 224}{2} = 216 \text{ is een oplossing}$$

$$v^3 = 208 - u^3 = 208 - 216 = -8$$

$$z = \sqrt[3]{216} + \sqrt[3]{-8} = 6 - 2 = 4$$

$$\begin{array}{r} z - 4 \mid z^3 \\ \underline{z^3 - 4z^2} \\ 4z^2 + 36z \\ \underline{4z^2 - 16z} \\ 52z - 208 \\ \underline{52z - 208} \\ 0 \end{array}$$

$$z^2 + 4z + 52 = 0$$

$$(z + 2)^2 - 4 + 52 = 0$$

$$(z + 2)^2 = -48$$

$$(z + 2)^2 = 48i^2$$

$$z + 2 = 4i\sqrt{3} \vee z + 2 = -4i\sqrt{3}$$

$$z = -2 + 4i\sqrt{3} \vee z = -2 - 4i\sqrt{3}$$

$$\text{Dus } z = 4 \vee z = -2 + 4i\sqrt{3} \vee z = -2 - 4i\sqrt{3}.$$

- b** $z = u + v$ en $-18 = -3uv$ geeft $u^3 + v^3 = 108$ $\left\{ \begin{array}{l} u^3 + \left(\frac{6}{u}\right)^3 = 108 \\ u^3 + \frac{216}{u^3} = 108 \end{array} \right.$

Uit $-3uv = -18$ volgt $v = \frac{6}{u}$.

$$u^3 + \frac{216}{u^3} = 108$$

$$u^6 - 108u^3 + 216 = 0$$

$$D = 10800, \text{ dus } \sqrt{D} = 60\sqrt{3}$$

$$u^3 = \frac{108 + 60\sqrt{3}}{2} = 54 + 30\sqrt{3} \text{ is een oplossing}$$

$$v^3 = 108 - u^3 = 108 - (54 + 30\sqrt{3}) = 54 - 30\sqrt{3}$$

$$z = \sqrt[3]{54 + 30\sqrt{3}} + \sqrt[3]{54 - 30\sqrt{3}} = 6$$

$$\begin{array}{r} z - 6 \mid z^3 \\ \underline{z^3 - 6z^2} \\ 6z^2 - 18z \\ \underline{6z^2 - 36z} \\ 18z - 108 \\ \underline{18z - 108} \\ 0 \end{array}$$

$$z^2 + 6z + 18 = 0$$

$$(z + 3)^2 - 9 + 18 = 0$$

$$(z + 3)^2 = -9$$

$$(z + 3)^2 = 9i^2$$

$$z + 3 = 3i \vee z + 3 = -3i$$

$$z = -3 + 3i \vee z = -3 - 3i$$

$$\text{Dus } z = 6 \vee z = -3 + 3i \vee z = -3 - 3i.$$

c $z = u + v$ en $2\frac{1}{4} = -3uv$ geeft $u^3 + v^3 = 3\frac{1}{4}$
 Uit $-3uv = 2\frac{1}{4}$ volgt $v = -\frac{3}{4u}$.

$$\left. \begin{aligned} u^3 + \left(-\frac{3}{4u}\right)^3 &= 3\frac{1}{4} \\ u^3 - \frac{27}{64u^3} &= 3\frac{1}{4} \\ u^6 - 3\frac{1}{4}u^3 - \frac{27}{64} &= 0 \\ D &= 12\frac{1}{4}, \text{ dus } \sqrt{D} = 3\frac{1}{2} \\ u^3 &= \frac{3\frac{1}{4} + 3\frac{1}{2}}{2} = 3\frac{3}{8} \text{ is een oplossing} \\ v^3 &= 3\frac{1}{4} - u^3 = 3\frac{1}{4} - 3\frac{3}{8} = -\frac{1}{8} \\ z &= \sqrt[3]{3\frac{3}{8}} + \sqrt[3]{-\frac{1}{8}} = 1\frac{1}{2} - \frac{1}{2} = 1 \end{aligned} \right\}$$

$$\begin{array}{r} z - 1 \quad / \quad z^3 \quad + 2\frac{1}{4}z - 3\frac{1}{4} \setminus z^2 + z + 3\frac{1}{4} \\ \underline{z^3 - z^2} \phantom{+ 2\frac{1}{4}z - 3\frac{1}{4}} \\ z^2 + 2\frac{1}{4}z \phantom{- 3\frac{1}{4}} \\ \underline{z^2 - z} \phantom{- 3\frac{1}{4}} \\ 3\frac{1}{4}z - 3\frac{1}{4} \\ \underline{3\frac{1}{4}z - 3\frac{1}{4}} \\ 0 \end{array}$$

$$z^2 + z + 3\frac{1}{4} = 0$$

$$(z + \frac{1}{2})^2 - \frac{1}{4} + 3\frac{1}{4} = 0$$

$$(z + \frac{1}{2})^2 = -3$$

$$(z + \frac{1}{2})^2 = 3i^2$$

$$z + \frac{1}{2} = i\sqrt{3} \vee z + \frac{1}{2} = -i\sqrt{3}$$

$$z = -\frac{1}{2} + i\sqrt{3} \vee z = -\frac{1}{2} - i\sqrt{3}$$

$$\text{Dus } z = 1 \vee z = -\frac{1}{2} + i\sqrt{3} \vee z = -\frac{1}{2} - i\sqrt{3}.$$

d $z = u + v$ en $81 = -3uv$ geeft $u^3 + v^3 = 702$
 Uit $-3uv = 81$ volgt $v = -\frac{27}{u}$.

$$\left. \begin{aligned} u^3 + \left(-\frac{27}{u}\right)^3 &= 702 \\ u^3 - \frac{19683}{u^3} &= 702 \\ u^6 - 702u^3 - 19683 &= 0 \\ D &= 5\,71\,536, \text{ dus } \sqrt{D} = 756 \\ u^3 &= \frac{702 + 756}{2} = 729 \text{ is een oplossing} \\ v^3 &= 702 - u^3 = 702 - 729 = -27 \\ z &= \sqrt[3]{729} + \sqrt[3]{-27} = 9 - 3 = 6 \end{aligned} \right\}$$

$$\begin{array}{r} z - 6 \quad / \quad z^3 \quad + 81z - 702 \setminus z^2 + 6z + 117 \\ \underline{z^3 - 6z^2} \\ 6z^2 + 81z \\ \underline{6z^2 - 36z} \\ 117z - 702 \\ \underline{117z - 702} \\ 0 \end{array}$$

$$z^2 + 6z + 117 = 0$$

$$(z + 3)^2 - 9 + 117 = 0$$

$$(z + 3)^2 = -108$$

$$(z + 3)^2 = 108i^2$$

$$z + 3 = 6i\sqrt{3} \vee z + 3 = -6i\sqrt{3}$$

$$z = -3 + 6i\sqrt{3} \vee z = -3 - 6i\sqrt{3}$$

$$\text{Dus } z = 6 \vee z = -3 + 6i\sqrt{3} \vee z = -3 - 6i\sqrt{3}.$$

Bladzijde 184

50 a $z^3 - 15z = 4$ ofwel $z^3 - 15z - 4 = 0$

Voer in $y_1 = x^3 - 15x - 4$.

De optie zero (TI) of ROOT (Casio) geeft $x = 4$.

Dus $z = 4$ is de gehele oplossing.

b $u^6 - 4u^3 + 125 = 0$

Stel $u^3 = x$.

$$x^2 - 4x + 125 = 0$$

$$(x - 2)^2 - 4 + 125 = 0$$

$$(x - 2)^2 = -121$$

$$(x - 2)^2 = 121i^2$$

$$x - 2 = 11i \vee x - 2 = -11i$$

$$x = 2 + 11i \vee x = 2 - 11i$$

$$u^3 = 2 + 11i \vee u^3 = 2 - 11i$$

Dus $u = \sqrt[3]{2 + 11i}$ is een oplossing.

$z = u + v$ en $-15 = -3uv$ geeft $u^3 + v^3 = 4$, dus $v^3 = 4 - u^3 = 4 - (2 + 11i) = 2 - 11i$.

Dus $v = \sqrt[3]{2 - 11i}$.

c De GR geeft $\sqrt[3]{2 + 11i} + \sqrt[3]{2 - 11i} = 4$.

Dus je krijgt de oplossing $z = 4$.

Bladzijde 186

51 a Substitutie van $z = y - \frac{1}{3}a$ in $z^3 + az^2 + bz + c = 0$ geeft

$$(y - \frac{1}{3}a)^3 + a(y - \frac{1}{3}a)^2 + b(y - \frac{1}{3}a) + c = 0$$

$$y^3 - ay^2 + \frac{1}{3}a^2y - \frac{1}{27}a^3 + a(y^2 - \frac{2}{3}ay + \frac{1}{9}a^2) + by - \frac{1}{3}ab + c = 0$$

$$y^3 - ay^2 + \frac{1}{3}a^2y - \frac{1}{27}a^3 + ay^2 - \frac{2}{3}a^2y + \frac{1}{9}a^3 + by - \frac{1}{3}ab + c = 0$$

$$y^3 + by - \frac{1}{3}a^2y + \frac{2}{27}a^3 - \frac{1}{3}ab + c = 0$$

$$y^3 + (b - \frac{1}{3}a^2)y = -\frac{2}{27}a^3 + \frac{1}{3}ab - c$$

b Stel $p = b - \frac{1}{3}a^2$ en $q = -\frac{2}{27}a^3 + \frac{1}{3}ab - c$.

Dit geeft $y^3 + py = q$.

Stel $y = u + v$ en $p = -3uv$.

Uit $p = -3uv$ volgt $v = -\frac{p}{3u}$.

$$u^3 + \left(-\frac{p}{3u}\right)^3 = q$$

$$u^3 - \frac{p^3}{27u^3} = q$$

$$u^3 - qu^3 - \frac{1}{27}p^3 = 0$$

$$D = q^2 + \frac{4}{27}p^3 = 4\left(\left(\frac{1}{2}q\right)^2 + \left(\frac{1}{3}p\right)^3\right), \text{ dus } \sqrt{D} = 2\sqrt{\left(\frac{1}{2}q\right)^2 + \left(\frac{1}{3}p\right)^3}$$

$$u^3 = \frac{q + 2\sqrt{\left(\frac{1}{2}q\right)^2 + \left(\frac{1}{3}p\right)^3}}{2} = \frac{1}{2}q + \sqrt{\left(\frac{1}{2}q\right)^2 + \left(\frac{1}{3}p\right)^3} \text{ is een oplossing.}$$

$$v^3 = q - u^3 = q - \left(\frac{1}{2}q + \sqrt{\left(\frac{1}{2}q\right)^2 + \left(\frac{1}{3}p\right)^3}\right) = \frac{1}{2}q - \sqrt{\left(\frac{1}{2}q\right)^2 + \left(\frac{1}{3}p\right)^3}$$

$$y = u + v = \sqrt[3]{\frac{1}{2}q + \sqrt{\left(\frac{1}{2}q\right)^2 + \left(\frac{1}{3}p\right)^3}} + \sqrt[3]{\frac{1}{2}q - \sqrt{\left(\frac{1}{2}q\right)^2 + \left(\frac{1}{3}p\right)^3}}$$

$$z = y - \frac{1}{3}a = -\frac{1}{3}a + \sqrt[3]{\frac{1}{2}q + \sqrt{\left(\frac{1}{2}q\right)^2 + \left(\frac{1}{3}p\right)^3}} + \sqrt[3]{\frac{1}{2}q - \sqrt{\left(\frac{1}{2}q\right)^2 + \left(\frac{1}{3}p\right)^3}}$$

Dus $z = -\frac{1}{3}a + \sqrt[3]{\frac{1}{2}q + \sqrt{\left(\frac{1}{2}q\right)^2 + \left(\frac{1}{3}p\right)^3}} + \sqrt[3]{\frac{1}{2}q - \sqrt{\left(\frac{1}{2}q\right)^2 + \left(\frac{1}{3}p\right)^3}}$ met $p = b - \frac{1}{3}a^2$ en

$q = -\frac{2}{27}a^3 + \frac{1}{3}ab - c$ is een reële oplossing van $z^3 + az^2 + bz + c = 0$.

52 a $z^3 + 6z^2 + 12z + 9 = 0$

$a = 6$, $b = 12$ en $c = 9$

$$p = b - \frac{1}{3}a^2 = 12 - \frac{1}{3} \cdot 6^2 = 0 \text{ en } q = -\frac{2}{27}a^3 + \frac{1}{3}ab - c = -\frac{2}{27} \cdot 6^3 + \frac{1}{3} \cdot 6 \cdot 12 - 9 = -1$$

$$z = -\frac{1}{3}a + \sqrt[3]{\frac{1}{2}q + \sqrt{\left(\frac{1}{2}q\right)^2 + \left(\frac{1}{3}p\right)^3}} + \sqrt[3]{\frac{1}{2}q - \sqrt{\left(\frac{1}{2}q\right)^2 + \left(\frac{1}{3}p\right)^3}} = -\frac{1}{3} \cdot 6 + \sqrt[3]{-\frac{1}{2} + \sqrt{\frac{1}{4} + 0}} + \sqrt[3]{-\frac{1}{2} - \sqrt{\frac{1}{4} + 0}} = -3$$

$$\begin{array}{r}
 z + 3 / z^3 + 6z^2 + 12z + 9 \setminus z^2 + 3z + 3 \\
 \underline{z^3 + 3z^2} \\
 3z^2 + 12z \\
 \underline{3z^2 + 9z} \\
 3z + 9 \\
 \underline{3z + 9} \\
 0
 \end{array}$$

$$\begin{aligned}
 z^2 + 3z + 3 &= 0 \\
 (z + 1\frac{1}{2})^2 - 2\frac{1}{4} + 3 &= 0 \\
 (z + 1\frac{1}{2})^2 &= -\frac{3}{4} \\
 (z + 1\frac{1}{2})^2 &= \frac{3}{4}i^2
 \end{aligned}$$

$$\begin{aligned}
 z + 1\frac{1}{2} &= \frac{1}{2}i\sqrt{3} \vee z + 1\frac{1}{2} = -\frac{1}{2}i\sqrt{3} \\
 z &= -1\frac{1}{2} + \frac{1}{2}i\sqrt{3} \vee z = -1\frac{1}{2} - \frac{1}{2}i\sqrt{3}
 \end{aligned}$$

Dus $z = -3 \vee z = -1\frac{1}{2} + \frac{1}{2}i\sqrt{3} \vee z = -1\frac{1}{2} - \frac{1}{2}i\sqrt{3}$.

b $z^3 - 12z^2 - 392 = 0$

$a = -12, b = 0$ en $c = -392$

$p = b - \frac{1}{3}a^2 = 0 - \frac{1}{3} \cdot (-12)^2 = -48$ en $q = -\frac{2}{27}a^3 + \frac{1}{3}ab - c = -\frac{2}{27} \cdot (-12)^3 + \frac{1}{3} \cdot -12 \cdot 0 - -392 = 520$

$$\begin{aligned}
 z &= -\frac{1}{3}a + \sqrt[3]{\frac{1}{2}q + \sqrt{(\frac{1}{2}q)^2 + (\frac{1}{3}p)^3}} + \sqrt[3]{\frac{1}{2}q - \sqrt{(\frac{1}{2}q)^2 + (\frac{1}{3}p)^3}} = \\
 &= -\frac{1}{3} \cdot -12 + \sqrt[3]{260 + \sqrt{260^2 + (-16)^3}} + \sqrt[3]{260 - \sqrt{260^2 + (-16)^3}} = 14
 \end{aligned}$$

$$\begin{array}{r}
 z - 14 / z^3 - 12z^2 - 392 \setminus z^2 + 2z + 28 \\
 \underline{z^3 - 14z^2} \\
 2z^2 \\
 \underline{2z^2 - 28z} \\
 28z - 392 \\
 \underline{28z - 392} \\
 0
 \end{array}$$

$$\begin{aligned}
 z^2 + 2z + 28 &= 0 \\
 (z + 1)^2 - 1 + 28 &= 0 \\
 (z + 1)^2 &= -27 \\
 (z + 1)^2 &= 27i^2 \\
 z + 1 &= 3i\sqrt{3} \vee z + 1 = -3i\sqrt{3} \\
 z &= -1 + 3i\sqrt{3} \vee z = -1 - 3i\sqrt{3} \\
 \text{Dus } z &= 14 \vee z = -1 + 3i\sqrt{3} \vee z = -1 - 3i\sqrt{3}.
 \end{aligned}$$

c $z^3 + 3z^2 + 4z - 28 = 0$

$a = 3, b = 4$ en $c = -28$

$p = b - \frac{1}{3}a^2 = 4 - \frac{1}{3} \cdot 3^2 = 1$ en $q = -\frac{2}{27}a^3 + \frac{1}{3}ab - c = -\frac{2}{27} \cdot 3^3 + \frac{1}{3} \cdot 3 \cdot 4 - -28 = 30$

$$\begin{aligned}
 z &= -\frac{1}{3}a + \sqrt[3]{\frac{1}{2}q + \sqrt{(\frac{1}{2}q)^2 + (\frac{1}{3}p)^3}} + \sqrt[3]{\frac{1}{2}q - \sqrt{(\frac{1}{2}q)^2 + (\frac{1}{3}p)^3}} = \\
 &= -\frac{1}{3} \cdot 3 + \sqrt[3]{15 + \sqrt{15^2 + (\frac{1}{3})^3}} + \sqrt[3]{15 - \sqrt{15^2 + (\frac{1}{3})^3}} = 2
 \end{aligned}$$

$$\begin{array}{r}
 z - 2 / z^3 + 3z^2 + 4z - 28 \setminus z^2 + 5z + 14 \\
 \underline{z^3 - 2z^2} \\
 5z^2 + 4z \\
 \underline{5z^2 - 10z} \\
 14z - 28 \\
 \underline{14z - 28} \\
 0
 \end{array}$$

$$\begin{aligned}
 z^2 + 5z + 14 &= 0 \\
 (z + 2\frac{1}{2})^2 - 6\frac{1}{4} + 14 &= 0 \\
 (z + 2\frac{1}{2})^2 &= -7\frac{3}{4} \\
 (z + 2\frac{1}{2})^2 &= \frac{31}{4}i^2 \\
 z + 2\frac{1}{2} &= \frac{1}{2}i\sqrt{31} \vee z + 2\frac{1}{2} = -\frac{1}{2}i\sqrt{31} \\
 z &= -2\frac{1}{2} + \frac{1}{2}i\sqrt{31} \vee z = -2\frac{1}{2} - \frac{1}{2}i\sqrt{31} \\
 \text{Dus } z &= 2 \vee z = -2\frac{1}{2} + \frac{1}{2}i\sqrt{31} \vee z = -2\frac{1}{2} - \frac{1}{2}i\sqrt{31}.
 \end{aligned}$$

- 53 a** De inhoud is 48, dus er moet gelden $abc = 48$.

De totale oppervlakte van de zijvlakken is 124, dus er moet gelden $2ab + 2ac + 2bc = 124$.

De lichaamsdiagonalen hebben lengte 101, dus er moet gelden $a^2 + b^2 + c^2 = 101$.

Dus om de lengte van de ribben a , b en c van de balk te berekenen moet het stelsel

$$\begin{cases} abc = 48 \\ 2ab + 2ac + 2bc = 124 \text{ worden opgelost.} \\ a^2 + b^2 + c^2 = 101 \end{cases}$$

- b** Als de lengte van één van de ribben gelijk is aan x geldt $x = a \vee x = b \vee x = c$

ofwel $x - a = 0 \vee x - b = 0 \vee x - c = 0$.

Hieruit volgt $(x - a)(x - b)(x - c) = 0$.

Dus als de lengte van één van de ribben gelijk is aan x geldt

$$(x - a)(x - b)(x - c) = 0$$

$$(x^2 - bx - ax + ab)(x - c) = 0$$

$$x^3 - cx^2 - bx^2 + bcx - ax^2 + acx + abx - abc = 0$$

$$x^3 - (a + b + c)x^2 + (ab + ac + bc)x - abc = 0$$

- c** $(a + b + c)^2 = (a + b + c)(a + b + c) = a^2 + ab + ac + ab + b^2 + bc + ac + bc + c^2 =$

$$a^2 + b^2 + c^2 + 2ab + 2ac + 2bc = 101 + 124 = 225$$

- d** $(a + b + c)^2 = 225$ geeft $a + b + c = 15$

$$2ab + 2ac + 2bc = 124 \text{ geeft } ab + ac + bc = 62$$

$$abc = 48$$

Dit geeft de vergelijking $x^3 - 15x^2 + 62x - 48 = 0$.

$$a = -15, b = 62 \text{ en } c = -48$$

$$p = b - \frac{1}{3}a^2 = 62 - \frac{1}{3} \cdot (-15)^2 = -13 \text{ en } q = -\frac{2}{27}a^3 + \frac{1}{3}ab - c = -\frac{2}{27} \cdot (-15)^3 + \frac{1}{3} \cdot -15 \cdot 62 - (-48) = -12$$

$$x = -\frac{1}{3}a + \sqrt[3]{\frac{1}{2}q + \sqrt{\left(\frac{1}{2}q\right)^2 + \left(\frac{1}{3}p\right)^3}} + \sqrt[3]{\frac{1}{2}q - \sqrt{\left(\frac{1}{2}q\right)^2 + \left(\frac{1}{3}p\right)^3}} =$$

$$-\frac{1}{3} \cdot -15 + \sqrt[3]{-6 + \sqrt{(-6)^2 + \left(-\frac{13}{3}\right)^3}} + \sqrt[3]{-6 - \sqrt{(-6)^2 + \left(-\frac{13}{3}\right)^3}} = 8$$

$$x - 8 \mid x^3 - 15x^2 + 62x - 48 \searrow x^2 - 7x + 6$$

$$\begin{array}{r} x^3 - 8x^2 \\ \underline{-7x^2 + 62x} \\ -7x^2 + 56x \\ \underline{6x - 48} \\ 6x - 48 \\ \underline{0} \end{array}$$

$$x^2 - 7x + 6 = 0$$

$$(x - 1)(x - 6) = 0$$

$$x = 1 \vee x = 6$$

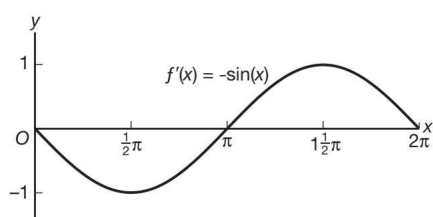
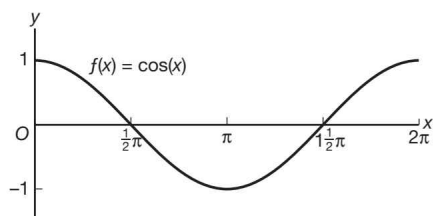
Dus $x = 1 \vee x = 6 \vee x = 8$.

De afmetingen van de balk zijn 8 bij 6 bij 1.

16.5 Wisselspanning

Bladzijde 188

- 54 a** $f(x) = \cos(x)$ geeft $f'(x) = -\sin(x)$



- b** De grafiek van $f'(x) = -\sin(x)$ ontstaat uit de grafiek van $f(x) = \cos(x)$ door de translatie $(-\frac{1}{2}\pi, 0)$.

- 55 a** $f(t) = 6 \cos(100\pi t)$ geeft $f'(t) = -600\pi \sin(100\pi t)$
 $= -600\pi \cos(100\pi t - \frac{1}{2}\pi)$
 $= 600\pi \cos(100\pi t - \frac{1}{2}\pi + \pi)$
 $= 600\pi \cos(100\pi t + \frac{1}{2}\pi)$
- b** $i = e^{\frac{1}{2}\pi i}$, dus $i \cdot 600\pi e^{i \cdot 100\pi t} = e^{\frac{1}{2}\pi i} \cdot 600\pi e^{i \cdot 100\pi t} = 600\pi e^{i \cdot 100\pi t + \frac{1}{2}\pi i} = 600\pi e^{i \cdot (100\pi t + \frac{1}{2}\pi)}$
- c** $600\pi e^{i \cdot (100\pi t + \frac{1}{2}\pi)} = 600\pi (\cos(100\pi t + \frac{1}{2}\pi) + i \sin(100\pi t + \frac{1}{2}\pi)) = 600\pi \cos(100\pi t + \frac{1}{2}\pi) + 600\pi i \sin(100\pi t + \frac{1}{2}\pi)$
 $\operatorname{Re}(g'(t)) = \operatorname{Re}(600\pi e^{i \cdot (100\pi t + \frac{1}{2}\pi)})$
 $= \operatorname{Re}(600\pi \cos(100\pi t + \frac{1}{2}\pi) + 600\pi i \sin(100\pi t + \frac{1}{2}\pi))$
 $= 600\pi \cos(100\pi t + \frac{1}{2}\pi)$
 $= f'(t)$

Bladzijde 190

- 56 a** $U = (2 + 2i\sqrt{3}) \cdot e^{i \cdot 100\pi t} = 4e^{\frac{1}{3}\pi i} \cdot e^{i \cdot 100\pi t} = 4e^{i \cdot (100\pi t + \frac{1}{3}\pi)}$
 De reële spanning is $\operatorname{Re}(U) = 4 \cos(100\pi t + \frac{1}{3}\pi)$.
- b** $U = (4 + 4i) \cdot e^{i \cdot 5t} = 4\sqrt{2}e^{\frac{1}{4}\pi i} \cdot e^{i \cdot 5t} = 4\sqrt{2}e^{i \cdot (5t + \frac{1}{4}\pi)}$
 De reële spanning is $\operatorname{Re}(U) = 4\sqrt{2} \cos(5t + \frac{1}{4}\pi)$.
- c** $U = (1 - i) \cdot e^{i \cdot 10t + 2} = \sqrt{2}e^{-\frac{1}{4}\pi i} \cdot e^{i \cdot 10t} \cdot e^2 = e^2 \sqrt{2}e^{i \cdot (10t - \frac{1}{4}\pi)}$
 De reële spanning is $\operatorname{Re}(U) = e^2 \sqrt{2} \cos(10t - \frac{1}{4}\pi)$.
- 57 a** $U = (10 + 10i) \cdot e^{i \cdot 20\pi t} = 10\sqrt{2}e^{\frac{1}{4}\pi i} \cdot e^{i \cdot 20\pi t} = 10\sqrt{2}e^{i \cdot (20\pi t + \frac{1}{4}\pi)}$
 $\arg(U) - \arg(I) = 20\pi t + \frac{1}{4}\pi - 20\pi t = \frac{1}{4}\pi$
 Het faseverschil is $\frac{1}{4}\pi$ rad.
- b** $\frac{U_{\max}}{I_{\max}} = \frac{10\sqrt{2}}{2} = 5\sqrt{2}$
- 58 a** $U = 12 \cos(ct)$ met $c = 2\pi f = 2\pi \cdot 100 = 200\pi$.
 Dus $U = 12 \cos(200\pi t)$.
- b** $U = 12(\cos(200\pi t) + i \sin(200\pi t)) = 12e^{i \cdot 200\pi t}$
- c** $I_1 = \frac{12e^{i \cdot 200\pi t}}{4} = 3e^{i \cdot 200\pi t}$
 $I_2 = \frac{12e^{i \cdot 200\pi t}}{6} = 2e^{i \cdot 200\pi t}$
- d** $I = 3e^{i \cdot 200\pi t} + 2e^{i \cdot 200\pi t} = 5e^{i \cdot 200\pi t}$

Bladzijde 191

- 59 a** Als de condensator volledig is opgeladen, verandert U niet meer.
 Verder is C een constante.
 Dus volgens $Q = C \cdot U$ verandert Q dan niet meer.
 Dus $I = \frac{dQ}{dt} = 0$.
 Er loopt dus geen stroom meer vanaf het moment dat de condensator volledig is opgeladen.
- b** $U = 220(\cos(100\pi t) + i \sin(100\pi t)) = 220e^{i \cdot 100\pi t}$
- c** $Q = C \cdot U$ geeft $0,002 \cdot 220e^{i \cdot 100\pi t} = 0,44e^{i \cdot 100\pi t}$
 $I = \frac{dQ}{dt} = 44\pi i e^{i \cdot 100\pi t}$

Bladzijde 193

- 60 a** $I_1 = \frac{U}{R} = \frac{50e^{i \cdot 30\pi t}}{3} = 16\frac{2}{3}e^{i \cdot 30\pi t}$
 $Q = C \cdot U = 0,05e^{i \cdot 30\pi t}$ geeft $I_2 = \frac{dQ}{dt} = 1,5\pi i e^{i \cdot 30\pi t}$
 $I = I_1 + I_2 = 16\frac{2}{3}e^{i \cdot 30\pi t} + 1,5\pi i e^{i \cdot 30\pi t} = (16\frac{2}{3} + 1,5\pi i)e^{i \cdot 30\pi t}$
- b** $Z = \frac{U_{\max}}{I_{\max}} = \frac{50}{|16\frac{2}{3} + 1,5\pi i|} = \frac{50}{\sqrt{(16\frac{2}{3})^2 + (1,5\pi)^2}} \approx 2,89 \Omega$
- c** $\arg(I) - \arg(U) = \arg(16\frac{2}{3} + 1,5\pi i) + 30\pi t - 30\pi t = \arg(16\frac{2}{3} + 1,5\pi i) \approx 0,28$ rad
 I loopt 0,28 rad voor op U .

61 $U = U_{\max} \cdot e^{i \cdot \omega t}$ geeft $\frac{dU}{dt} = \omega \cdot U_{\max} \cdot e^{i \cdot (\omega t + \frac{1}{2}\pi)}$

$Q = C \cdot U$ met C een constante geeft $\frac{dQ}{dt} = C \cdot \frac{dU}{dt}$.

Dus $I = \frac{dQ}{dt} = C \cdot \frac{dU}{dt} = C \cdot \omega \cdot U_{\max} \cdot e^{i \cdot (\omega t + \frac{1}{2}\pi)}$ met $I_{\max} = C \cdot \omega \cdot U_{\max}$.

Dus $Z = \frac{U_{\max}}{I_{\max}} = \frac{U_{\max}}{C \cdot \omega \cdot U_{\max}} = \frac{1}{\omega C}$.

62 a Volgens de wet van Ohm is $U_R = I \cdot R = 6 e^{i \cdot 10\pi t} \cdot 5 = 30 e^{i \cdot 10\pi t}$.

b $\frac{dQ}{dt} = I = 6 e^{i \cdot 10\pi t}$ geeft $Q = \int 6 e^{i \cdot 10\pi t} dt = \frac{6}{10\pi i} e^{i \cdot 10\pi t} = -\frac{6}{10\pi} i e^{i \cdot 10\pi t}$

Uit $Q = C \cdot U$ volgt $U_C = \frac{Q}{C} = \frac{-\frac{6}{10\pi} i e^{i \cdot 10\pi t}}{0,01} = -\frac{60}{\pi} i e^{i \cdot 10\pi t}$.

c $U = U_R + U_C = 30 e^{i \cdot 10\pi t} - \frac{60}{\pi} i e^{i \cdot 10\pi t} = \left(30 - \frac{60}{\pi} i\right) e^{i \cdot 10\pi t}$

d $Z = \frac{U_{\max}}{I_{\max}} = \frac{\left|30 - \frac{60}{\pi} i\right|}{6} = \frac{\sqrt{30^2 + \left(-\frac{60}{\pi}\right)^2}}{6} \approx 5,93 \Omega$

e $\arg(U) - \arg(I) = \arg\left(30 - \frac{60}{\pi} i\right) + 10\pi t - 10\pi t = \arg\left(30 - \frac{60}{\pi} i\right) \approx -0,57$
 I loopt 0,57 rad voor op U .

Bladzijde 194

63 a Stel $U = U_{\max} e^{i \cdot \omega t}$.

Voor de condensator geldt $Q = C \cdot U = C \cdot U_{\max} e^{i \cdot \omega t}$.

De stroom door de condensator is $I_1 = \frac{dQ}{dt} = C \cdot i\omega \cdot U_{\max} e^{i \cdot \omega t}$.

De stroom door de weerstand is $I_2 = \frac{U}{R} = \frac{U_{\max} e^{i \cdot \omega t}}{R}$.

$I = I_1 + I_2 = C \cdot i\omega \cdot U_{\max} e^{i \cdot \omega t} + \frac{U_{\max} e^{i \cdot \omega t}}{R}$

$\frac{1}{Z} = \frac{I_{\max}}{U_{\max}} = \frac{\left|C \cdot i\omega \cdot U_{\max} e^{i \cdot \omega t} + \frac{U_{\max} e^{i \cdot \omega t}}{R}\right|}{\left|U_{\max} e^{i \cdot \omega t}\right|} = \frac{\left|U_{\max} e^{i \cdot \omega t}\right| \cdot \left|C \cdot i\omega + \frac{1}{R}\right|}{\left|U_{\max} e^{i \cdot \omega t}\right|} = \left|C \cdot i\omega + \frac{1}{R}\right| = \sqrt{(\omega C)^2 + \left(\frac{1}{R}\right)^2}$

b Stel $I = I_{\max} e^{i \cdot \omega t}$.

Voor de spanning tussen de platen van de condensator geldt $\frac{dQ}{dt} = I = I_{\max} e^{i \cdot \omega t}$, dus

$Q = \frac{I_{\max} e^{i \cdot \omega t}}{i\omega}$ en $U = \frac{I_{\max} e^{i \cdot \omega t}}{i\omega C}$.

De spanning tussen de uiteinden van de weerstand is $U = I \cdot R = I_{\max} e^{i \cdot \omega t} \cdot R = I_{\max} R e^{i \cdot \omega t}$.

$U = \frac{I_{\max} e^{i \cdot \omega t}}{i\omega C} + I_{\max} R e^{i \cdot \omega t}$

$Z = \frac{U_{\max}}{I_{\max}} = \frac{\left|\frac{I_{\max} e^{i \cdot \omega t}}{i\omega C} + I_{\max} R e^{i \cdot \omega t}\right|}{\left|I_{\max} e^{i \cdot \omega t}\right|} = \frac{\left|I_{\max} e^{i \cdot \omega t}\right| \cdot \left|\frac{1}{i\omega C} + R\right|}{\left|I_{\max} e^{i \cdot \omega t}\right|} = \left|\frac{1}{i\omega C} + R\right| = \sqrt{\left(\frac{1}{\omega C}\right)^2 + R^2}$

64 a $\frac{dI}{dt} = 5$ en $L = 0,012$ geeft $U_{\text{spoel}} = L \cdot \frac{dI}{dt} = 0,012 \cdot 5 = 0,06$ volt.

b $\omega = 2\pi f = 2\pi \cdot 50 = 100\pi$ rad/s en $U_{\max} = 220$ V, dus $U = 220 e^{i \cdot 100\pi t}$.

$U = L \cdot \frac{dI}{dt}$ ofwel $\frac{dI}{dt} = \frac{U}{L}$ met $U = 220 e^{i \cdot 100\pi t}$ en $L = 0,005$ geeft $\frac{dI}{dt} = \frac{220 e^{i \cdot 100\pi t}}{0,005} = 44000 e^{i \cdot 100\pi t}$.

Dus $I = \frac{44000}{100\pi i} e^{i \cdot 100\pi t} = -\frac{440}{\pi} i e^{i \cdot 100\pi t}$.

$Z = \frac{U_{\max}}{I_{\max}} = \frac{220}{\left|-\frac{440}{\pi} i\right|} = \frac{220}{\left(\frac{440}{\pi}\right)} = \frac{1}{2}\pi \Omega$

c $\arg(I) - \arg(U) = \arg\left(-\frac{440}{\pi}i\right) + 100\pi t - 100\pi t = \arg\left(-\frac{440}{\pi}i\right) = -\frac{1}{2}\pi \text{ rad}$
 U loopt $\frac{1}{2}\pi \text{ rad}$ voor op I .

Bladzijde 195

65 a Uit $U = U_{\max} \cdot e^{i\omega t}$ en $I = \frac{U_{\max}}{\omega L} \cdot e^{i \cdot (\omega t - \frac{1}{2}\pi)}$ volgt $\arg(U) - \arg(I) = \omega t - (\omega t - \frac{1}{2}\pi) = \frac{1}{2}\pi \text{ rad}$.
 U loopt $\frac{1}{2}\pi \text{ rad}$ voor op I .
 b $Z = \frac{U_{\max}}{I_{\max}} = \frac{U_{\max}}{\left(\frac{U_{\max}}{\omega L}\right)} = \omega L$

Bladzijde 196

66 De stroom op $t = 0$ is $I = \frac{U_{\max}}{\omega L} \cdot e^{i \cdot (0 - \frac{1}{2}\pi)} = \frac{U_{\max}}{\omega L} \cdot e^{-\frac{1}{2}\pi i} = -\frac{U_{\max}}{\omega L} i$.
 Het reële deel hiervan is 0, dus is er geen stroom op $t = 0$.

67 a Voor de weerstand geldt $U_R = I \cdot R = 2e^{i \cdot 100\pi t} \cdot 20 = 40e^{i \cdot 100\pi t}$.
 Voor de spoel geldt $U_{\text{spoel}} = L \cdot \frac{dI}{dt} = 0,1 \cdot 200\pi i e^{i \cdot 100\pi t} = 20\pi i e^{i \cdot 100\pi t}$.
 $U = U_R + U_{\text{spoel}} = 40e^{i \cdot 100\pi t} + 20\pi i e^{i \cdot 100\pi t} = (40 + 20\pi i)e^{i \cdot 100\pi t}$

b $Z = \frac{U_{\max}}{I_{\max}} = \frac{|40 + 20\pi i|}{2} = \frac{\sqrt{40^2 + (20\pi)^2}}{2} \approx 37,2 \Omega$

c $\arg(U) - \arg(I) = \arg(40 + 20\pi i) + 100\pi t - 100\pi t = \arg(40 + 20\pi i) \approx 1,00 \text{ rad}$
 U loopt 1,00 rad voor op I .

68 a Voor de condensator geldt $Q = C \cdot U = 0,001 \cdot 50e^{i \cdot 1000\pi t} = 0,05e^{i \cdot 1000\pi t}$ en $I_C = \frac{dQ}{dt} = 50\pi i e^{i \cdot 1000\pi t}$.
 Voor de spoel geldt $I_{\text{spoel}} = -i \cdot \frac{U_{\max}}{\omega L} \cdot e^{i\omega t} = -i \cdot \frac{50}{1000\pi \cdot 0,0005} \cdot e^{i\omega t} = -\frac{100}{\pi} i e^{i\omega t}$.
 Dus $I = I_C + I_{\text{spoel}} = 50\pi i e^{i \cdot 1000\pi t} - \frac{100}{\pi} i e^{i \cdot 1000\pi t} = \left(50\pi - \frac{100}{\pi}\right) i e^{i \cdot 1000\pi t}$.

b $Z = \frac{U_{\max}}{I_{\max}} = \frac{50}{50\pi - \frac{100}{\pi}} \approx 0,399 \Omega$

c $\arg(I) - \arg(U) = \arg\left(\left(50\pi - \frac{100}{\pi}\right)i\right) + 1000\pi t - 1000\pi t = \frac{1}{2}\pi + 1000\pi t - 1000\pi t = \frac{1}{2}\pi \text{ rad}$
 I loopt $\frac{1}{2}\pi \text{ rad}$ voor op U .

69 a Voor de stroom I_1 geldt $I_1 = \frac{dQ}{dt} = C \cdot \frac{dU}{dt} = C \cdot U_{\max} \cdot i\omega e^{i\omega t}$.
 Voor de stroom I_2 geldt $\frac{dI_2}{dt} = \frac{U}{L} = \frac{U_{\max} \cdot e^{i\omega t}}{L} = \frac{U_{\max}}{L} e^{i\omega t}$, dus $I_2 = \frac{U_{\max}}{i\omega L} e^{i\omega t} = -\frac{U_{\max}}{\omega L} i e^{i\omega t}$.
 Hieruit volgt $I = I_1 + I_2 = C \cdot U_{\max} \cdot i\omega e^{i\omega t} - \frac{U_{\max}}{\omega L} i e^{i\omega t} = U_{\max} \left(\omega C - \frac{1}{\omega L}\right) i e^{i\omega t} =$
 $U_{\max} \left(\omega C - \frac{1}{\omega L}\right) e^{\frac{1}{2}\pi i} \cdot e^{i\omega t} = U_{\max} \left(\omega C - \frac{1}{\omega L}\right) e^{i \cdot (\omega t + \frac{1}{2}\pi)}$.

b $I = 0$ geeft $U_{\max} \left(\omega C - \frac{1}{\omega L}\right) e^{i \cdot (\omega t + \frac{1}{2}\pi)} = 0$
 $\omega C - \frac{1}{\omega L} = 0$
 $\omega C = \frac{1}{\omega L}$
 $\omega^2 = \frac{1}{LC}$
 $\omega = \frac{1}{\sqrt{LC}}$

Bladzijde 197

70 Voor de weerstand geldt $I_R = \frac{U_{\max} \cdot e^{i\omega t}}{R}$.

Voor de condensator geldt $I_C = C \cdot U_{\max} \cdot i\omega e^{i\omega t}$ (zie opgave 69).

Voor de spoel geldt $I_{\text{spoel}} = -\frac{U_{\max}}{\omega L} i e^{i\omega t}$ (zie opgave 69).

Dus $I = I_R + I_C + I_{\text{spoel}} = \frac{U_{\max} \cdot e^{i\omega t}}{R} + C \cdot U_{\max} \cdot i\omega e^{i\omega t} - \frac{U_{\max}}{\omega L} i e^{i\omega t} = U_{\max} \left(\frac{1}{R} + \left(\omega C - \frac{1}{\omega L} \right) i \right) e^{i\omega t}$.

$$Z = \frac{U_{\max}}{I_{\max}} = \frac{U_{\max}}{U_{\max} \left| \frac{1}{R} + \left(\omega C - \frac{1}{\omega L} \right) i \right|} = \frac{1}{\left| \frac{1}{R} + \left(\omega C - \frac{1}{\omega L} \right) i \right|} = \frac{1}{\sqrt{\left(\frac{1}{R} \right)^2 + \left(\omega C - \frac{1}{\omega L} \right)^2}}$$

De impedantie is maximaal als $\sqrt{\left(\frac{1}{R} \right)^2 + \left(\omega C - \frac{1}{\omega L} \right)^2}$ minimaal is, dus als $\left(\omega C - \frac{1}{\omega L} \right)^2 = 0$ ofwel

$$\omega C - \frac{1}{\omega L} = 0. \text{ Hieruit volgt } \omega = \frac{1}{\sqrt{LC}}.$$

Dus voor $\omega = \frac{1}{\sqrt{LC}}$ is de impedantie maximaal.

71 $96,8 \text{ MHz} = 96,8 \cdot 10^6 \text{ Hz}$
 $\omega = 2\pi f = 2\pi \cdot 96,8 \cdot 10^6 = 193,6\pi \cdot 10^6 \text{ rad/s}$ en $L = 6,76 \cdot 10^{-9}$

Uit $\omega = \frac{1}{\sqrt{LC}}$ volgt $\sqrt{LC} = \frac{1}{\omega}$

$$LC = \frac{1}{\omega^2}$$

$$C = \frac{1}{L\omega^2}$$

Dit geeft $C = \frac{1}{6,76 \cdot 10^{-9} \cdot (193,6\pi \cdot 10^6)^2} \approx 4 \cdot 10^{-10} \text{ F}$.

De condensator moet worden ingesteld op 400 pF ($1 \text{ pF} = 10^{-12} \text{ F}$).

Diagnostische toets**Bladzijde 200**

1 a $f(2+i) = 0$ geeft $2(2+i) + a + bi = 0$
 $4 + 2i + a + bi = 0$
 $a + bi = -4 - 2i$

Dus $a = -4$ en $b = -2$.

b $f(2+i) = 2+i$ geeft $2(2+i) + a + bi = 2+i$
 $4 + 2i + a + bi = 2+i$
 $a + bi = -2-i$

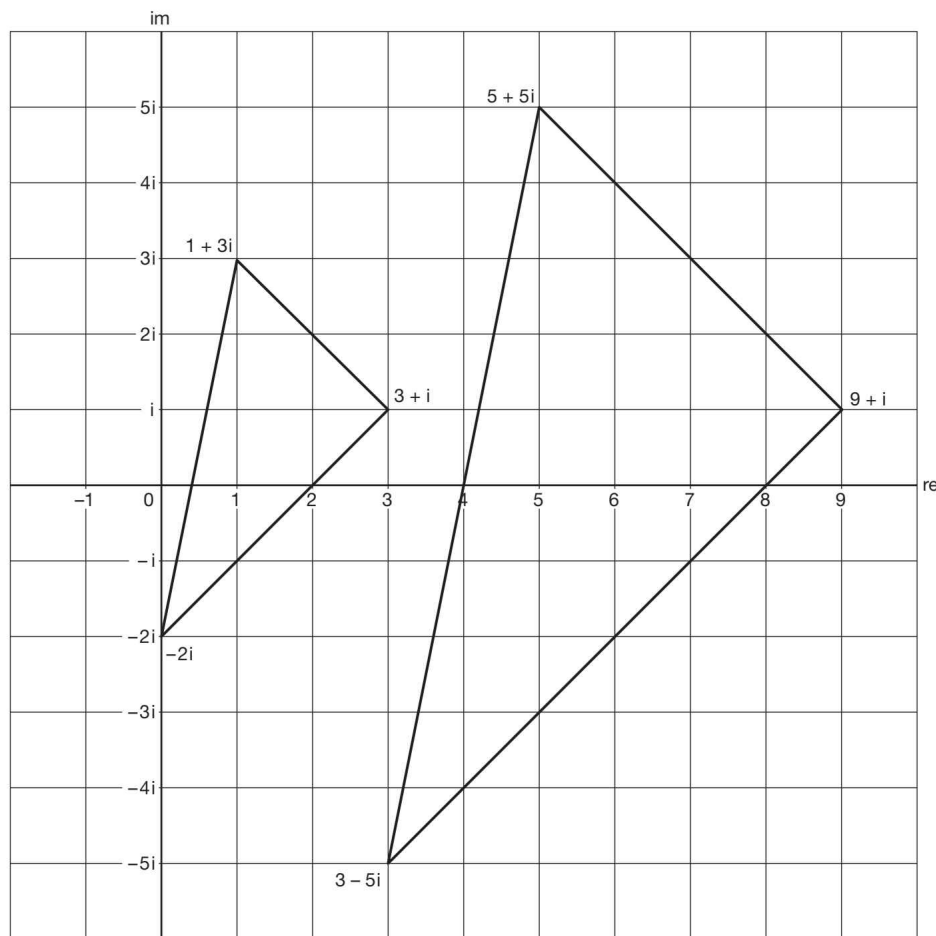
Dus $a = -2$ en $b = -1$.

c $a = 3$ en $b = -1$ geeft $f(z) = 2z + 3 - i$

$$f(-2i) = 3 - 5i$$

$$f(3 + i) = 9 + i$$

$$f(1 + 3i) = 5 + 5i$$



2 a $f(z) = 0$ geeft $2iz + 3 - i = 0$

$$2iz = -3 + i$$

$$z = \frac{1}{2} - \frac{3}{2i} = \frac{1}{2} + 1\frac{1}{2}i$$

Het nulpunt is $\frac{1}{2} + 1\frac{1}{2}i$.

$$f(z) = z \text{ geeft } 2iz + 3 - i = z$$

$$2iz - z = -3 + i$$

$$(2i - 1)z = -3 + i$$

$$z = \frac{-3 + i}{2i - 1} \cdot \frac{2i + 1}{2i + 1} = \frac{-6i - 3 + 2i^2 + i}{4i^2 - 1} = \frac{-5 - 5i}{-5} = 1 + i$$

Het dekpunt is $1 + i$.

b $g(z) = 0$ geeft $(1 + i)z + 2 - 3i = 0$

$$(1 + i)z = -2 + 3i$$

$$z = \frac{-2 + 3i}{1 + i} \cdot \frac{1 - i}{1 - i} = \frac{-2 + 2i + 3i - 3i^2}{1 - i^2} = \frac{1 + 5i}{2} = \frac{1}{2} + 2\frac{1}{2}i$$

Het nulpunt is $\frac{1}{2} + 2\frac{1}{2}i$.

$$g(z) = z \text{ geeft } (1 + i)z + 2 - 3i = z$$

$$iz = -2 + 3i$$

$$z = 3 - \frac{2}{i} = 3 + 2i$$

Het dekpunt is $3 + 2i$.

- 3 a** $|2 + 2i\sqrt{3}| = 4$ en $\text{Arg}(2 + 2i\sqrt{3}) = \frac{1}{3}\pi$, dus rotatie om 0 over $\frac{1}{3}\pi$ en vermenigvuldiging ten opzichte van 0 met 4.

Het bereik is $z \geq 8 \wedge \frac{5}{6}\pi \leq \text{Arg}(z) \leq 1\frac{1}{3}\pi$.

- b** De lijnen $\text{Re}(z) = 2$ en $\text{Im}(z) = 2\sqrt{3}$ snijden elkaar in $2 + 2i\sqrt{3}$.

$$f(2 + 2i\sqrt{3}) = (2 + 2i\sqrt{3})(2 + 2i\sqrt{3}) = 4 + 8i\sqrt{3} - 12 = -8 + 8i\sqrt{3}$$

2 ligt op $\text{Re}(z) = 2$.

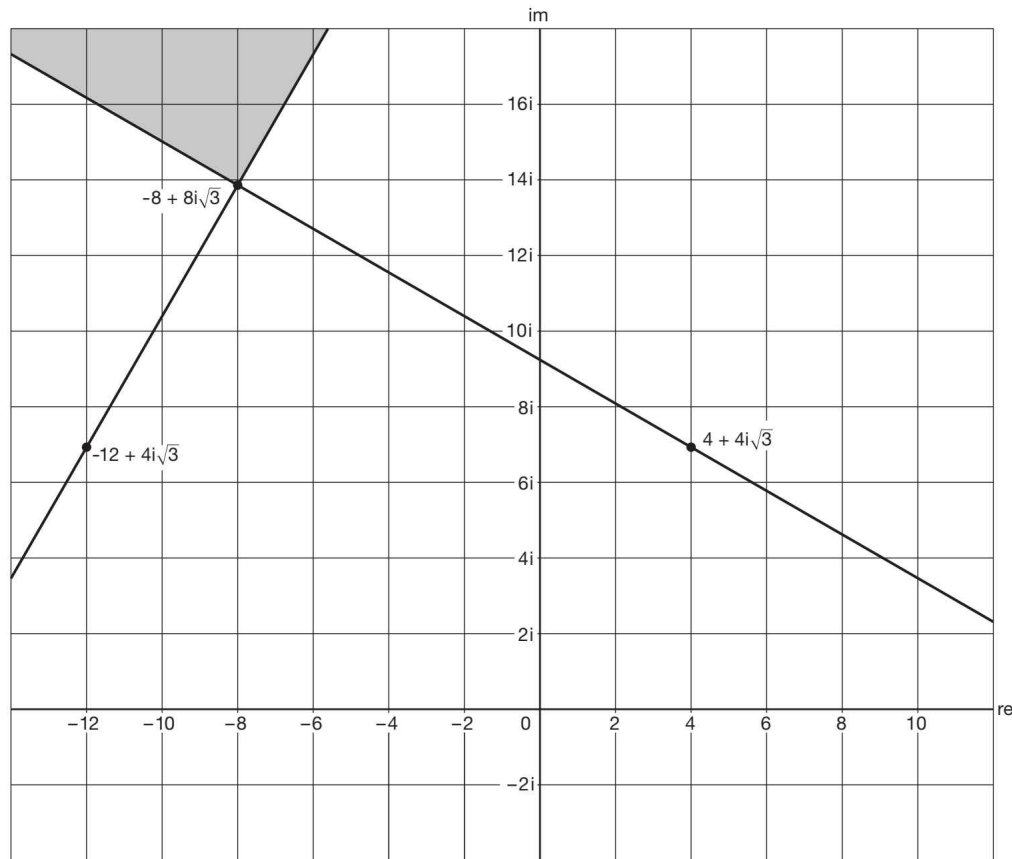
$$f(2) = (4 + 4i\sqrt{3}) \cdot 2 = 4 + 4i\sqrt{3}$$

Het beeld van $\text{Re}(z) = 2$ is de lijn door $-8 + 8i\sqrt{3}$ en $4 + 4i\sqrt{3}$.

$2i\sqrt{3}$ ligt op $\text{Im}(z) = 2\sqrt{3}$.

$$f(2i\sqrt{3}) = (2 + 2i\sqrt{3}) \cdot 2i\sqrt{3} = -12 + 4i\sqrt{3}$$

Het beeld van $\text{Im}(z) = 2\sqrt{3}$ is de lijn door $-8 + 8i\sqrt{3}$ en $-12 + 4i\sqrt{3}$.



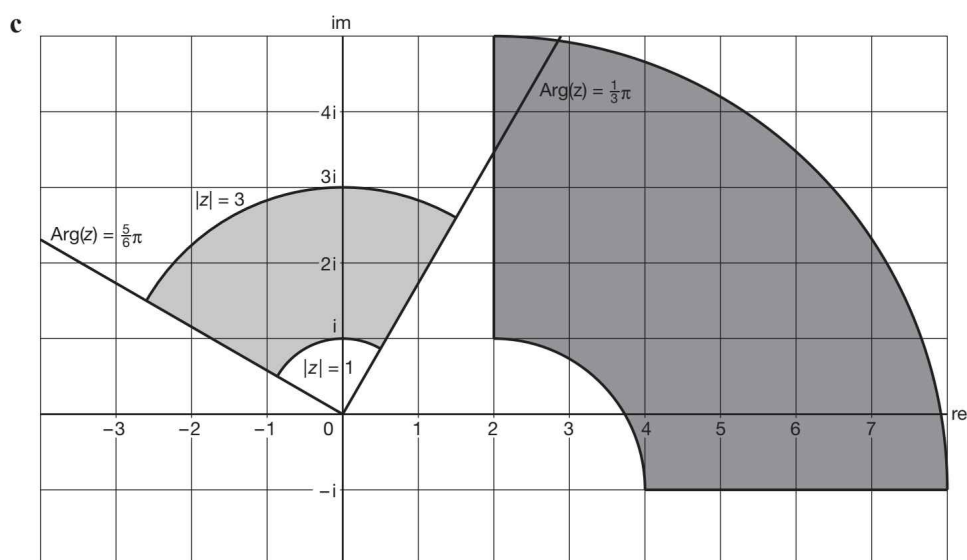
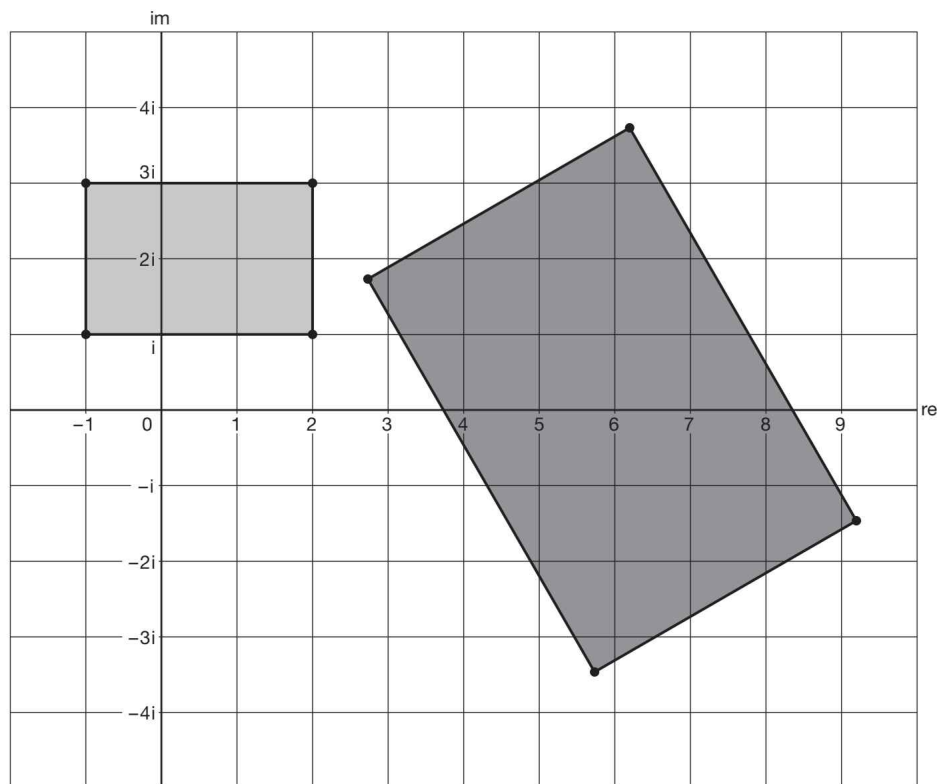
- 4 a** $f(z) = 2 + 2i$ geeft $(1 - i\sqrt{3})z + 2 - i = 2 + 2i$

$$(1 - i\sqrt{3})z = 3i$$

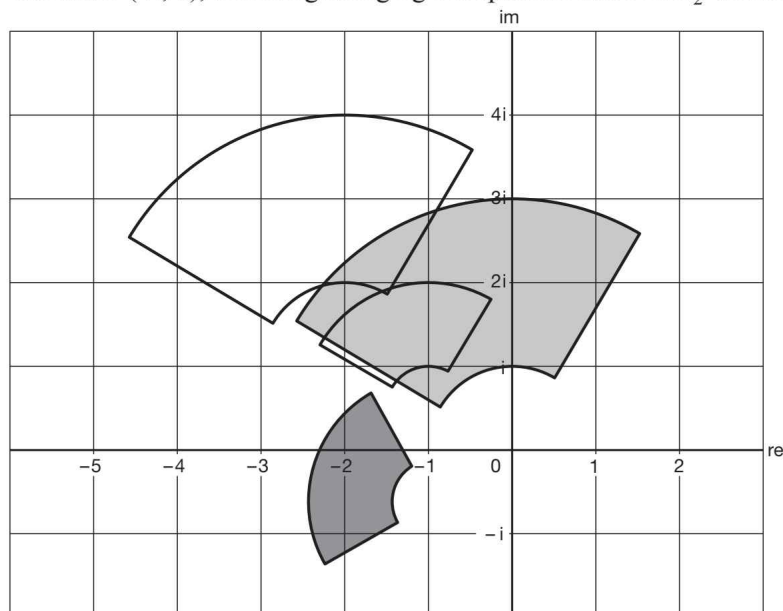
$$z = \frac{3i}{1 - i\sqrt{3}} \cdot \frac{1 + i\sqrt{3}}{1 + i\sqrt{3}} = \frac{-3\sqrt{3} + 3i}{4} = -\frac{3}{4}\sqrt{3} + \frac{3}{4}i$$

Her origineel is $-\frac{3}{4}\sqrt{3} + \frac{3}{4}i$.

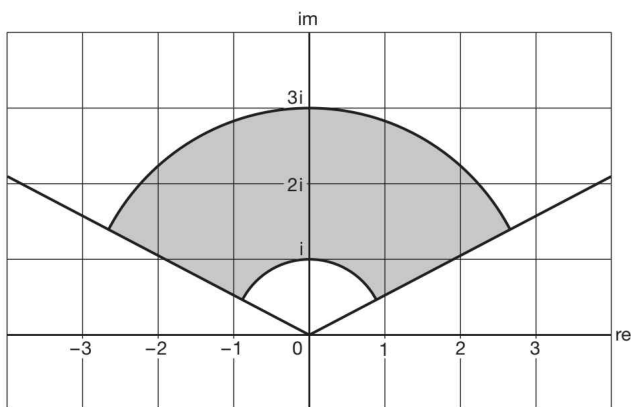
- b** $|1 - i\sqrt{3}| = 2$ en $\text{Arg}(1 - i\sqrt{3}) = -\frac{1}{3}\pi$, dus rotatie om 0 over $-\frac{1}{3}\pi$, vermenigvuldiging ten opzichte van 0 met 2 en translatie $(2, -1)$.



d Translatie $(-2, 1)$, vermenigvuldiging ten opzichte van 0 met $\frac{1}{2}$ en rotatie om 0 over $\frac{1}{3}\pi$.

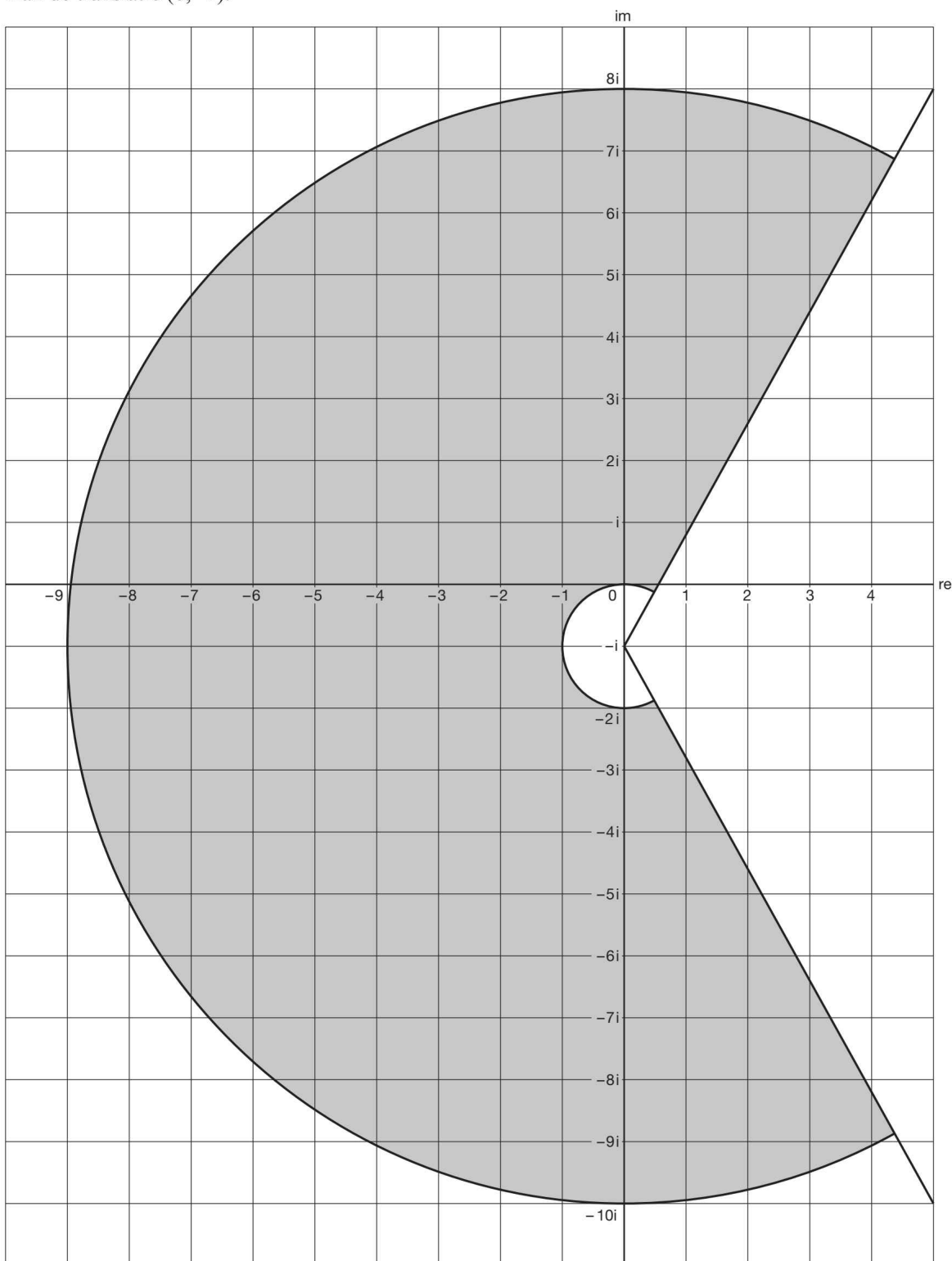


5



Eerst z^2 : $|z^2| = |z|^2$, dus $1 \leq |z^2| \leq 9$ en $\arg(z^2) = 2 \cdot \arg(z)$, dus $\frac{1}{3}\pi \leq \arg(z^2) \leq \frac{2}{3}\pi$.
 Dan de translatie $(0, -1)$.

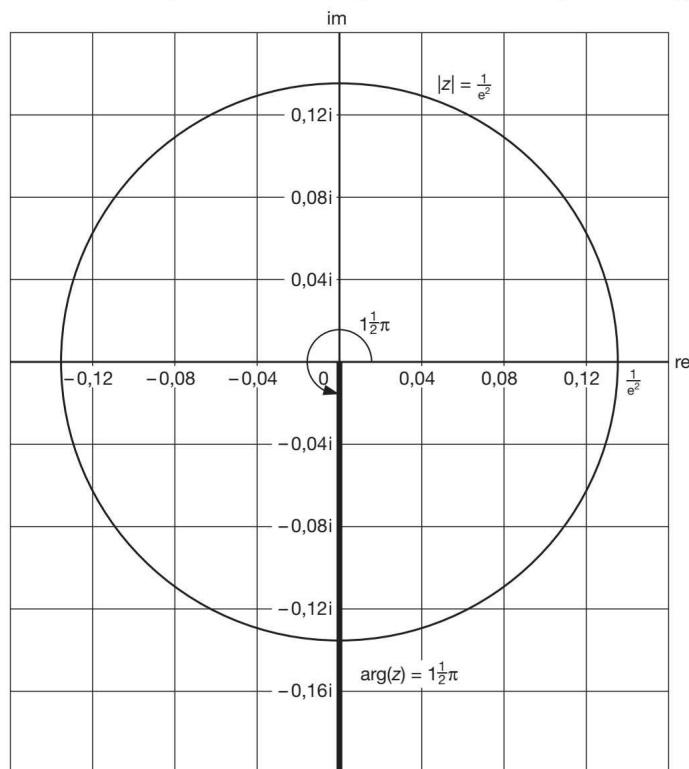
16



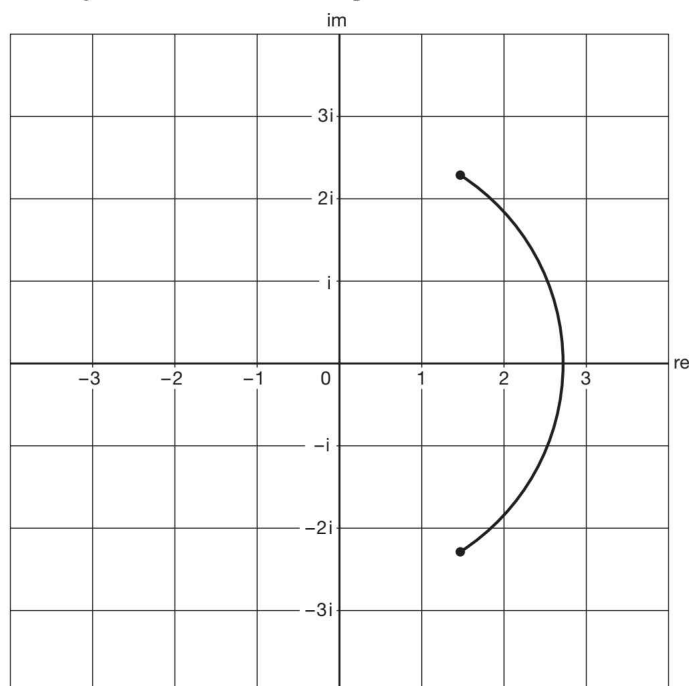
- 6 a** Het beeld van $\operatorname{Re}(z) = -2$ is de cirkel met middelpunt 0 en straal e^{-2} , ofwel de cirkel

met vergelijking $|z| = e^{-2} = \frac{1}{e^2}$.

Het beeld van $\operatorname{Im}(z) = 1\frac{1}{2}\pi$ is de halve lijn met beginpunt 0 die een hoek van $1\frac{1}{2}\pi$ radialen maakt met de positieve reële as, ofwel de halve lijn met vergelijking $\arg(z) = 1\frac{1}{2}\pi$.



- b** De eindpunten van het lijnstuk zijn $1 - i$ en $1 + i$, dus $\operatorname{Re}(z) = 1$ is vast. Het beeld van het lijnstuk is een deel van een cirkel met middelpunt 0 en straal e waarbij een hoek wordt doorlopen van -1 tot 1 radialen.



- 7 a** $f(1 + i) = \ln(1 + i) = \ln(\sqrt{2} e^{\frac{1}{4}\pi i}) = \ln(\sqrt{2}) + \ln(e^{\frac{1}{4}\pi i}) = \frac{1}{2} \ln(2) + \frac{1}{4}\pi i$
b $f\left(\frac{i}{e}\right) = \ln\left(\frac{i}{e}\right) = \ln(i) - \ln(e) = \ln(e^{\frac{1}{2}\pi i}) - 1 = -1 + \frac{1}{2}\pi i$
c $f(-e^3) = \ln(-e^3) = \ln(-1) + \ln(e^3) = \ln(e^{\pi i}) + 3 = 3 + \pi i$
d $f(i\sqrt{e}) = \ln(i\sqrt{e}) = \ln(i) + \ln(\sqrt{e}) = \ln(e^{\frac{1}{2}\pi i}) + \frac{1}{2} = \frac{1}{2} + \frac{1}{2}\pi i$

$$\begin{aligned} \text{e } f\left(\frac{2i}{1-i}\right) &= \ln\left(\frac{2i}{1-i}\right) = \ln(2) + \ln(i) - \ln(1-i) = \ln(2) + \ln(e^{\frac{1}{2}\pi i}) - \ln(\sqrt{2}e^{-\frac{1}{4}\pi i}) = \ln(2) + \frac{1}{2}\pi i - \ln(\sqrt{2}) - \ln(e^{-\frac{1}{4}\pi i}) = \\ &= \ln(2) + \frac{1}{2}\pi i - \frac{1}{2}\ln(2) + \frac{1}{4}\pi i = \frac{1}{2}\ln(2) + \frac{3}{4}\pi i \\ \text{f } f(e - ei\sqrt{3}) &= \ln(e - ei\sqrt{3}) = \ln(e(1 - i\sqrt{3})) = \ln(e) + \ln(1 - i\sqrt{3}) = 1 + \ln(2e^{-\frac{1}{3}\pi i}) = 1 + \ln(2) + \ln(e^{-\frac{1}{3}\pi i}) = \\ &= 1 + \ln(2) - \frac{1}{3}\pi i \end{aligned}$$

- 8 a** Bij een punt van c hoort het complexe getal $z = e^2 \cdot e^{i\varphi}$.
Dit geeft $f(e^2 \cdot e^{i\varphi}) = \ln(e^2 \cdot e^{i\varphi}) = \ln(e^2) + \ln(e^{i\varphi}) = 2 + i\varphi$.
Omdat is afgesproken dat $-\pi < \text{Im}(f(z)) \leq \pi$ is het beeld van c het lijnstuk met eindpunten $2 - \pi i$ en $2 + \pi i$.

- b** Voor het tekenen van het beeld van $\text{Im}(z) = \frac{1}{2}\pi$ bereken je een aantal functiewaarden.

$$f(-3 + \frac{1}{2}\pi i) \approx 1,22 + 2,66i$$

$$f(-2 + \frac{1}{2}\pi i) \approx 0,93 + 2,48i$$

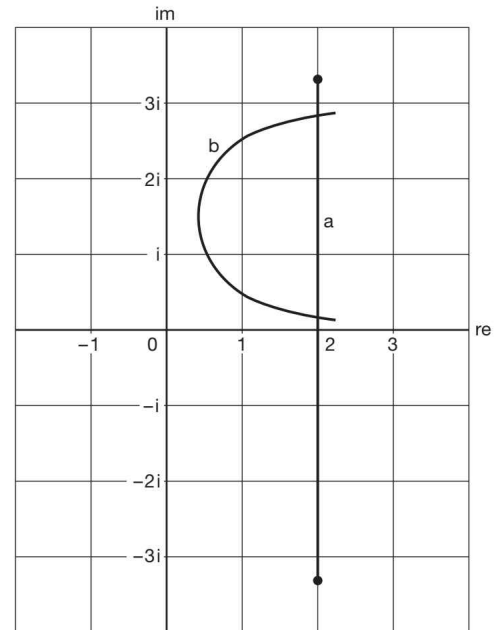
$$f(-1 + \frac{1}{2}\pi i) \approx 0,62 + 2,14i$$

$$f(\frac{1}{2}\pi i) \approx 0,45 + 1,57i$$

$$f(1 + \frac{1}{2}\pi i) \approx 0,62 + 1,00i$$

$$f(2 + \frac{1}{2}\pi i) \approx 0,93 + 0,67i$$

$$f(3 + \frac{1}{2}\pi i) \approx 1,22 + 0,48i$$



Bladzijde 201

$$\begin{aligned} \text{9 } f\left(\frac{1}{3}\pi + 2i\right) &= \sin\left(\frac{1}{3}\pi + 2i\right) = \frac{e^{i \cdot (\frac{1}{3}\pi + 2i)} - e^{-i \cdot (\frac{1}{3}\pi + 2i)}}{2i} = \frac{e^{-2 + \frac{1}{3}\pi i} - e^{2 - \frac{1}{3}\pi i}}{2i} = \frac{e^{-2} \cdot e^{\frac{1}{3}\pi i} - e^2 \cdot e^{-\frac{1}{3}\pi i}}{2i} = \\ &= \frac{e^{-2}(\frac{1}{2} + \frac{1}{2}i\sqrt{3}) - e^2(\frac{1}{2} - \frac{1}{2}i\sqrt{3})}{2i} = \frac{\frac{1}{2}e^{-2} + \frac{1}{2}e^{-2}i\sqrt{3}}{2i} - \frac{\frac{1}{2}e^2 + \frac{1}{2}e^2i\sqrt{3}}{2i} = \frac{-i}{4e^2} + \frac{\sqrt{3}}{4e^2} + \frac{ie^2}{4} + \frac{e^2\sqrt{3}}{4} = \\ &= \frac{\sqrt{3}}{4e^2} + \frac{e^2\sqrt{3}}{4} + \left(\frac{e^2}{4} - \frac{1}{4e^2}\right)i \\ g\left(\frac{1}{2}\pi - i\right) &= \cos\left(\frac{1}{2}\pi - i\right) = \frac{e^{i \cdot (\frac{1}{2}\pi - i)} + e^{-i \cdot (\frac{1}{2}\pi - i)}}{2} = \frac{e^{1 + \frac{1}{2}\pi i} + e^{-1 - \frac{1}{2}\pi i}}{2} = \frac{e^1 \cdot e^{\frac{1}{2}\pi i} + e^{-1} \cdot e^{-\frac{1}{2}\pi i}}{2} = \\ &= \frac{e \cdot i + e^{-1} \cdot -i}{2} = \left(\frac{e}{2} - \frac{1}{2e}\right)i \end{aligned}$$

$$\text{10 a } h = b + i(a - b) = ia + (1 - i)b$$

$$\text{b } e = b - i(c - b) = (1 + i)b - ic$$

$$\begin{aligned} \text{c } p &= b + (h - b) + (p - h) \\ &= b + (h - b) + (e - b) \\ &= b + i(a - b) + i(b - c) \\ &= ia + b - ic \end{aligned}$$

$$\text{d } g = c - i(a - c) = -ia + (1 + i)c$$

$$\text{e } d = c + i(b - c) = ib + (1 - i)c$$

$$\begin{aligned} \text{f } q &= c + (d - c) + (q - d) \\ &= c + (d - c) + (g - c) \\ &= c + i(b - c) + i(c - a) \\ &= -ia + ib + c \end{aligned}$$

$$\text{g } i(p - a) = i(ia + b - ic - a) = (-1 - i)a + ib + c$$

$$q - a = -ia + ib + c - a = (-1 - i)a + ib + c$$

$$\text{Dus } i(p - a) = q - a.$$

Hieruit volgt dat $AP = AQ$ en $AP \perp AQ$.

Dus driehoek APQ is een gelijkbenige rechthoekige driehoek met $AP = AQ$ en $\angle A = 90^\circ$.

- 11 a** Voer in $y_1 = x^3 + x^2 - 7x - 15$.

De optie zero (TI) of ROOT (Casio) geeft $x = 3$.

$$z - 3 \mid z^3 + z^2 - 7z - 15 \setminus z^2 + 4z + 5$$

$$\begin{array}{r} z^3 - 3z^2 \\ \hline 4z^2 - 7z \\ 4z^2 - 12z \\ \hline 5z - 15 \\ 5z - 15 \\ \hline 0 \end{array}$$

$$z^2 + 4z + 5 = 0$$

$$(z + 2)^2 - 4 + 5 = 0$$

$$(z + 2)^2 = -1$$

$$(z + 2)^2 = i^2$$

$$z + 2 = i \vee z + 2 = -i$$

$$z = -2 + i \vee z = -2 - i$$

$$\text{Dus } z = 3 \vee z = -2 + i \vee z = -2 - i.$$

- b** $(z + 1)(z - i)(z + i) = 15$

$$(z + 1)(z^2 - i^2) = 15$$

$$(z + 1)(z^2 + 1) = 15$$

$$z^3 + z^2 + z + 1 = 15$$

$$z^3 + z^2 + z - 14 = 0$$

$$\text{Voer in } y_1 = x^3 + x^2 + x - 14.$$

De optie zero (TI) of ROOT (Casio) geeft $x = 2$.

$$z - 2 \mid z^3 + z^2 + z - 14 \setminus z^2 + 3z + 7$$

$$\begin{array}{r} z^3 - 2z^2 \\ \hline 3z^2 + z \\ 3z^2 - 6z \\ \hline 7z - 14 \\ 7z - 14 \\ \hline 0 \end{array}$$

$$z^2 + 3z + 7 = 0$$

$$(z + 1\frac{1}{2})^2 - 2\frac{1}{4} + 7 = 0$$

$$(z + 1\frac{1}{2})^2 = -4\frac{3}{4}$$

$$(z + 1\frac{1}{2})^2 = \frac{19}{4}i^2$$

$$z + 1\frac{1}{2} = \frac{1}{2}i\sqrt{19} \vee z + 1\frac{1}{2} = -\frac{1}{2}i\sqrt{19}$$

$$z = -1\frac{1}{2} + \frac{1}{2}i\sqrt{19} \vee z = -1\frac{1}{2} - \frac{1}{2}i\sqrt{19}$$

$$\text{Dus } z = 2 \vee z = -1\frac{1}{2} + \frac{1}{2}i\sqrt{19} \vee z = -1\frac{1}{2} - \frac{1}{2}i\sqrt{19}.$$

12 a $z^3 + 6z - 88 = 0$ ofwel $z^3 + 6z = 88$

$$z = u + v \text{ en } 6 = -3uv \text{ geeft } u^3 + v^3 = 88$$

$$\left. \begin{array}{l} \text{Uit } -3uv = 6 \text{ volgt } v = -\frac{2}{u} \\ u^3 + \left(-\frac{2}{u}\right)^3 = 88 \end{array} \right\}$$

$$u^3 - \frac{8}{u^3} = 88$$

$$u^6 - 88u^3 - 8 = 0$$

$$D = 7776, \text{ dus } \sqrt{D} = 36\sqrt{6}$$

$$u^3 = \frac{88 + 36\sqrt{6}}{2} = 44 + 18\sqrt{6} \text{ is een oplossing}$$

$$v^3 = 88 - u^3 = 88 - (44 + 18\sqrt{6}) = 44 - 18\sqrt{6}$$

$$z = \sqrt[3]{44 + 18\sqrt{6}} + \sqrt[3]{44 - 18\sqrt{6}} = 4$$

$$\begin{array}{r} z - 4 / z^3 \quad + \quad 6z - 88 \setminus z^2 + 4z + 22 \\ \underline{z^3 - 4z^2} \quad - \\ 4z^2 + 6z \\ \underline{4z^2 - 16z} \quad - \\ 22z - 88 \\ \underline{22z - 88} \quad - \\ 0 \end{array}$$

$$z^2 + 4z + 22 = 0$$

$$(z + 2)^2 - 4 + 22 = 0$$

$$(z + 2)^2 = -18$$

$$(z + 2)^2 = 18i^2$$

$$z + 2 = 3i\sqrt{2} \vee z + 2 = -3i\sqrt{2}$$

$$z = -2 + 3i\sqrt{2} \vee z = -2 - 3i\sqrt{2}$$

$$\text{Dus } z = 4 \vee z = -2 + 3i\sqrt{2} \vee z = -2 - 3i\sqrt{2}.$$

b $z^3 - 3z^2 = 16$ ofwel $z^3 - 3z^2 - 16 = 0$

$$a = -3, b = 0 \text{ en } c = -16$$

$$p = b - \frac{1}{3}a^2 = 0 - \frac{1}{3} \cdot (-3)^2 = -3 \text{ en } q = -\frac{2}{27}a^3 + \frac{1}{3}ab - c = -\frac{2}{27} \cdot (-3)^3 + \frac{1}{3} \cdot -3 \cdot 0 - -16 = 18$$

$$z = -\frac{1}{3}a + \sqrt[3]{\frac{1}{2}q + \sqrt{\left(\frac{1}{2}q\right)^2 + \left(\frac{1}{3}p\right)^3}} + \sqrt[3]{\frac{1}{2}q - \sqrt{\left(\frac{1}{2}q\right)^2 + \left(\frac{1}{3}p\right)^3}} =$$

$$-\frac{1}{3} \cdot -3 + \sqrt[3]{9 + \sqrt{9^2 + (-1)^3}} + \sqrt[3]{9 - \sqrt{9^2 + (-1)^3}} = 4$$

$$z - 4 / z^3 - 3z^2 \quad - 16 \setminus z^2 + z + 4$$

$$\begin{array}{r} \underline{z^3 - 4z^2} \quad - \\ z^2 \\ \underline{z^2 - 4z} \quad - \\ 4z - 16 \\ \underline{4z - 16} \quad - \\ 0 \end{array}$$

$$z^2 + z + 4 = 0$$

$$\left(z + \frac{1}{2}\right)^2 - \frac{1}{4} + 4 = 0$$

$$\left(z + \frac{1}{2}\right)^2 = -3\frac{3}{4}$$

$$\left(z + \frac{1}{2}\right)^2 = \frac{15}{4}i^2$$

$$z + \frac{1}{2} = \frac{1}{2}i\sqrt{15} \vee z + \frac{1}{2} = -\frac{1}{2}i\sqrt{15}$$

$$z = -\frac{1}{2} + \frac{1}{2}i\sqrt{15} \vee z = -\frac{1}{2} - \frac{1}{2}i\sqrt{15}$$

$$\text{Dus } z = 4 \vee z = -\frac{1}{2} + \frac{1}{2}i\sqrt{15} \vee z = -\frac{1}{2} - \frac{1}{2}i\sqrt{15}.$$

c $z^3 + 4z^2 + 6z + 4 = 0$

$a = 4, b = 6$ en $c = 4$

$p = b - \frac{1}{3}a^2 = 6 - \frac{1}{3} \cdot 4^2 = \frac{2}{3}$ en $q = -\frac{2}{27}a^3 + \frac{1}{3}ab - c = -\frac{2}{27} \cdot 4^3 + \frac{1}{3} \cdot 4 \cdot 6 - 4 = -\frac{20}{27}$

$z = -\frac{1}{3}a + \sqrt[3]{\frac{1}{2}q + \sqrt{\left(\frac{1}{2}q\right)^2 + \left(\frac{1}{3}p\right)^3}} + \sqrt[3]{\frac{1}{2}q - \sqrt{\left(\frac{1}{2}q\right)^2 + \left(\frac{1}{3}p\right)^3}} =$
 $-\frac{1}{3} \cdot 4 + \sqrt[3]{-\frac{10}{27} + \sqrt{\left(-\frac{10}{27}\right)^2 + \left(\frac{2}{9}\right)^3}} + \sqrt[3]{-\frac{10}{27} - \sqrt{\left(-\frac{10}{27}\right)^2 + \left(\frac{2}{9}\right)^3}} = -2$

$z + 2 \mid z^3 + 4z^2 + 6z + 4 \setminus z^2 + 2z + 2$

$$\begin{array}{r} z^3 + 2z^2 \\ \hline 2z^2 + 6z \\ \hline 2z^2 + 4z \\ \hline 2z + 4 \\ \hline 2z + 4 \\ \hline 0 \end{array}$$

$z^2 + 2z + 2 = 0$

$(z + 1)^2 - 1 + 2 = 0$

$(z + 1)^2 = -1$

$(z + 1)^2 = i^2$

$z + 1 = i \vee z + 1 = -i$

$z = -1 + i \vee z = -1 - i$

Dus $z = -2 \vee z = -1 + i \vee z = -1 - i$.

13 a $\omega = 2\pi f = 2\pi \cdot 500 = 1000\pi \text{ rad/s}$

$U = U_{\max} \cdot e^{i \cdot \omega t} = 360 e^{i \cdot 1000\pi t}$

b $I_1 = \frac{360 e^{i \cdot 1000\pi t}}{0,4} = 900 e^{i \cdot 1000\pi t}$

$I_2 = \frac{360 e^{i \cdot 1000\pi t}}{0,8} = 450 e^{i \cdot 1000\pi t}$

$I = I_1 + I_2 = 900 e^{i \cdot 1000\pi t} + 450 e^{i \cdot 1000\pi t} = 1350 e^{i \cdot 1000\pi t}$

14 a Voor de stroom I_1 geldt volgens de wet van Ohm $I_1 = \frac{U}{R} = \frac{U_{\max} \cdot e^{i\omega t}}{R} = \frac{10 e^{i \cdot 1000\pi t}}{500} = \frac{1}{50} e^{i \cdot 1000\pi t}$.

Zie opgave 69. Voor de stroom I_2 geldt $I_2 = C \cdot U_{\max} \cdot i\omega e^{i\omega t} = 0,001 \cdot 10 \cdot 1000\pi i e^{i \cdot 1000\pi t} = 10\pi i e^{i \cdot 1000\pi t}$.

Zie opgave 69. Voor de stroom I_3 geldt $I_3 = -\frac{U_{\max}}{\omega L} i e^{i\omega t} = -\frac{10}{1000\pi \cdot 0,01} i e^{i \cdot 1000\pi t} = -\frac{1}{\pi} i e^{i \cdot 1000\pi t}$.

Dus $I = I_1 + I_2 + I_3 = \frac{1}{50} e^{i \cdot 1000\pi t} + 10\pi i e^{i \cdot 1000\pi t} - \frac{1}{\pi} i e^{i \cdot 1000\pi t} = \left(\frac{1}{50} + \left(10\pi - \frac{1}{\pi}\right)i\right) e^{i \cdot 1000\pi t}$.

b $Z = \frac{U_{\max}}{I_{\max}} = \frac{10}{\left|\frac{1}{50} + \left(10\pi - \frac{1}{\pi}\right)i\right|} = \frac{10}{\sqrt{\left(\frac{1}{50}\right)^2 + \left(10\pi - \frac{1}{\pi}\right)^2}} \approx 0,32 \Omega$

c $\arg(I) - \arg(U) = \arg\left(\frac{1}{50} + \left(10\pi - \frac{1}{\pi}\right)i\right) + 1000\pi t - 1000\pi t = \arg\left(\frac{1}{50} + \left(10\pi - \frac{1}{\pi}\right)i\right) \approx 1,57 \text{ rad}$

I loopt 1,57 rad voor op U .