Gemengde opgaven

9 Exponentiële en logaritmische functies

Bladzijde 226

1 **a** $x + 1 > 0$ geeft $x > -1$, dus $D_f = \langle -1, \rightarrow\rangle$.

$f(x) = g(x)$ geeft $2 - {}^3\log(x+1) = {}^{\frac{1}{3}}\log(5-x)$

$${}^{\frac{1}{3}}\log(\tfrac{1}{9}) + {}^{\frac{1}{3}}\log(x+1) = {}^{\frac{1}{3}}\log(5-x)$$
$${}^{\frac{1}{3}}\log(\tfrac{1}{9}(x+1)) = {}^{\frac{1}{3}}\log(5-x)$$
$$\tfrac{1}{9}(x+1) = 5 - x$$
$$x + 1 = 45 - 9x$$
$$10x = 44$$
$$x = 4\tfrac{2}{5}$$

$f(x) \geq g(x)$ geeft $-1 < x \leq 4\frac{2}{5}$

b $f(x) = 3$ geeft $2 - {}^3\log(x+1) = 3$

$$-{}^3\log(x+1) = 1$$
$${}^3\log(x+1) = -1$$
$$x + 1 = 3^{-1}$$
$$x + 1 = \tfrac{1}{3}$$
$$x = -\tfrac{2}{3}$$

$g(x) = 3$ geeft ${}^{\frac{1}{3}}\log(5-x) = 3$

$$5 - x = (\tfrac{1}{3})^3$$
$$5 - x = \tfrac{1}{27}$$
$$x = 4\tfrac{26}{27}$$

$AB = 4\frac{26}{27} - -\frac{2}{3} = 4\frac{26}{27} + \frac{18}{27} = 5\frac{17}{27}$

$f(2) = 2 - {}^3\log(3) = 2 - 1 = 1$ en $g(2) = {}^{\frac{1}{3}}\log(3) = {}^{\frac{1}{3}}\log((\frac{1}{3})^{-1}) = -1$

$CD = 1 - -1 = 2$

Dus AB is niet meer dan drie keer zo lang als CD.

c $f(x) = 2 - {}^3\log(x+1)$ geeft $f'(x) = -\dfrac{1}{(x+1)\ln(3)}$

$\text{rc}_{\text{raaklijn}} = -1$, dus $f'(x) = -1$

$$-\frac{1}{(x+1)\ln(3)} = -1$$
$$\frac{1}{(x+1)\ln(3)} = 1$$
$$x + 1 = \frac{1}{\ln(3)}$$
$$x = \frac{1}{\ln(3)} - 1$$
$$x = \frac{1 - \ln(3)}{\ln(3)}$$

Dus $x_P = \dfrac{1 - \ln(3)}{\ln(3)}$.

2 **a**
$$4^{\log(x)} = 2^{3 + \log(x)}$$
$$(2^2)^{\log(x)} = 2^{3 + \log(x)}$$
$$2^{2\log(x)} = 2^{3 + \log(x)}$$
$$2\log(x) = 3 + \log(x)$$
$$\log(x) = 3$$
$$x = 1000$$

b
$${}^8\log(2x+1) = {}^4\log(25)$$
$$\frac{{}^2\log(2x+1)}{{}^2\log(8)} = \frac{{}^2\log(25)}{{}^2\log(4)}$$
$$\frac{{}^2\log(2x+1)}{3} = \frac{{}^2\log(25)}{2}$$
$${}^2\log(2x+1) = 1\tfrac{1}{2} \cdot {}^2\log(25)$$
$${}^2\log(2x+1) = {}^2\log(25^{1\frac{1}{2}})$$
$$2x + 1 = 125$$
$$2x = 124$$
$$x = 62$$

c $^5\log^2(x+2) = 6 \cdot {}^5\log(x+2) + 7$
Stel $^5\log(x+2) = u$.
$u^2 = 6u + 7$
$u^2 - 6u - 7 = 0$
$(u+1)(u-7) = 0$
$u = -1 \vee u = 7$
$^5\log(x+2) = -1 \vee {}^5\log(x+2) = 7$
$x + 2 = 5^{-1} \vee x + 2 = 5^7$
$x = -2 + \frac{1}{5} \vee x = 5^7 - 2$
$x = -1\frac{4}{5} \vee x = 78\,123$

d $2 \cdot {}^3\log(2x-3) + {}^{\frac{1}{3}}\log(2x+1) = 2$
$^3\log((2x-3)^2) - {}^3\log(2x+1) = {}^3\log(9)$
$^3\log((2x-3)^2) = {}^3\log(9) + {}^3\log(2x+1)$
$^3\log((2x-3)^2) = {}^3\log(9(2x+1))$
$(2x-3)^2 = 9(2x+1)$
$4x^2 - 12x + 9 = 18x + 9$
$4x^2 - 30x = 0$
$4x(x - 7\frac{1}{2}) = 0$
$x = 0 \quad \vee \quad x = 7\frac{1}{2}$
vold. niet

3 **a** $\ln^2(x) + 1 = 2\frac{1}{2}\ln(x)$
Stel $\ln(x) = u$.
$u^2 + 1 = 2\frac{1}{2}u$
$u^2 - 2\frac{1}{2}u + 1 = 0$
$(u - \frac{1}{2})(u - 2) = 0$
$u = \frac{1}{2} \vee u = 2$
$\ln(x) = \frac{1}{2} \vee \ln(x) = 2$
$x = \sqrt{e} \vee x = e^2$

b $\dfrac{e^x}{e^x - 2} = 2$
$e^x = 2(e^x - 2)$
$e^x = 2\,e^x - 4$
$-e^x = -4$
$e^x = 4$
$x = \ln(4)$

c $\ln(3x+2) = \frac{1}{2}$
$3x + 2 = e^{\frac{1}{2}}$
$3x + 2 = \sqrt{e}$
$3x = -2 + \sqrt{e}$
$x = -\frac{2}{3} + \frac{1}{3}\sqrt{e}$

d $\ln(4x) - \ln(x+4) = 1$
$\ln\left(\dfrac{4x}{x+4}\right) = 1$
$\dfrac{4x}{x+1} = e$
$4x = e(x+1)$
$4x = ex + 4e$
$4x - ex = 4e$
$(4 - e)x = 4e$
$x = \dfrac{4e}{4 - e}$

e $\ln^2(x-2) = 4$
$\ln(x-2) = 2 \vee \ln(x-2) = -2$
$x - 2 = e^2 \vee x - 2 = e^{-2}$
$x = 2 + e^2 \vee x = 2 + \dfrac{1}{e^2}$

f $3\,e^{2x} + 2 = 5\,e^x$
$3\,(e^x)^2 + 2 = 5\,e^x$
Stel $e^x = u$.
$3u^2 + 2 = 5u$
$3u^2 - 5u + 2 = 0$
$D = (-5^2) - 4 \cdot 3 \cdot 2 = 1$
$u = \dfrac{5+1}{6} = 1 \vee u = \dfrac{5-1}{6} = \frac{2}{3}$
$e^x = 1 \vee e^x = \frac{2}{3}$
$x = 0 \vee x = \ln(\frac{2}{3})$

4 **a** $g_{\text{jaar}} = 1{,}096$
$1{,}096^T = 2$
$T = {}^{1{,}096}\log(2) = 7{,}561...$
De verdubbelingstijd is 7 jaar en $0{,}561... \cdot 12 \approx 7$ maanden.

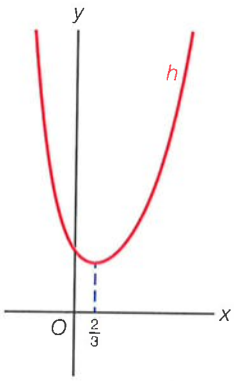
b $g_{\text{dag}} = 0{,}83$
$0{,}83^T = \frac{1}{2}$
$T = {}^{0{,}83}\log(\frac{1}{2}) = 3{,}720...$
De halveringstijd is $3{,}720... \cdot 24 \approx 89$ uren.

c $g_{\text{maand}} = 2$
$g_{\text{dag}} = 2^{\frac{1}{30}} = 1{,}023...$
De toename per dag is 2,3%.

d $g_{8{,}3 \text{ dagen}} = \frac{1}{2}$
$g_{\text{dag}} = (\frac{1}{2})^{\frac{1}{8{,}3}} = 0{,}919...$
$0{,}919...^t = 0{,}01$
$t = {}^{0{,}919...}\log(0{,}01) = 55{,}1...$
Dus na 55 dagen is er nog 1% over.

5 **a** $f(x) = g(x)$ geeft $3 \cdot 2^x = 6 \cdot (\frac{1}{4})^x$
$3 \cdot 2^x = 6 \cdot (2^{-2})^x$
$3 \cdot 2^x = 6 \cdot (2^x)^{-2}$
Stel $2^x = u$.
$3u = 6u^{-2}$
$3u^3 = 6$
$u^3 = 2$
$u = 2^{\frac{1}{3}}$
$2^x = 2^{\frac{1}{3}}$
$x = \frac{1}{3}$
$f(x) \leq g(x)$ geeft $x \leq \frac{1}{3}$

b $h(x) = f(x) + g(x) = 3 \cdot 2^x + 6 \cdot (\frac{1}{4})^x$ geeft $h'(x) = 3 \cdot 2^x \cdot \ln(2) + 6 \cdot (\frac{1}{4})^x \cdot \ln(\frac{1}{4})$
$h'(x) = 0$ geeft $3 \cdot 2^x \cdot \ln(2) + 6 \cdot (\frac{1}{4})^x \cdot \ln(\frac{1}{4}) = 0$
$3 \cdot 2^x \cdot \ln(2) + 6 \cdot (2^{-2})^x \cdot \ln(2^{-2}) = 0$
$3 \cdot 2^x \cdot \ln(2) + 6 \cdot 2^{-2x} \cdot -2\ln(2) = 0$
$3 \cdot 2^x \cdot \ln(2) - 12 \cdot 2^{-2x} \cdot \ln(2) = 0$
$3 \cdot 2^x \cdot \ln(2) = 12 \cdot 2^{-2x} \cdot \ln(2)$
$2^x = 4 \cdot 2^{-2x}$
$2^x = 2^2 \cdot 2^{-2x}$
$2^x = 2^{2-2x}$
$x = 2 - 2x$
$3x = 2$
$x = \frac{2}{3}$



Dus $h(x) = f(x) + g(x)$ heeft een minimum voor $x = \frac{2}{3}$.

Bladzijde 227

6 **a** $f(x) = x^2 e^{x-1}$ geeft $f'(x) = 2x \cdot e^{x-1} + x^2 \cdot e^{x-1} = (x^2 + 2x)e^{x-1}$

b $g(x) = \ln^2(x) + \ln(x^2) = (\ln(x))^2 + \ln(x^2)$ geeft
$g'(x) = 2\ln(x) \cdot \frac{1}{x} + \frac{1}{x^2} \cdot 2x = \frac{2\ln(x)}{x} + \frac{2}{x} = \frac{2\ln(x) + 2}{x}$

c $h(x) = {}^2\log(x^3 - x^2)$ geeft $h'(x) = \frac{1}{(x^3 - x^2)\ln(2)} \cdot (3x^2 - 2x) = \frac{3x^2 - 2x}{(x^3 - x^2)\ln(2)}$

d $j(x) = \ln(\ln(2x))$ geeft $j'(x) = \frac{1}{\ln(2x)} \cdot \frac{1}{2x} \cdot 2 = \frac{1}{x\ln(2x)}$

7 **a** $y = e^{3x-1} \xrightarrow{\text{translatie } (2,\, 0)} y = e^{3(x-2)-1}$
Er geldt $e^{3(x-2)-1} = e^{3x-6-1} = e^{3x-1-6} = e^{3x-1} \cdot e^{-6} = \frac{1}{e^6} \cdot e^{3x-1}$.
Dus de vermenigvuldiging ten opzichte van de x-as met $\frac{1}{e^6}$ levert dezelfde beeldfiguur op.

b $y = \ln(x^2) \xrightarrow{\text{verm. } y\text{-as, } 3} y = \ln((\frac{1}{3}x)^2) = \ln(\frac{1}{9}x^2)$
Er geldt $\ln(\frac{1}{9}x^2) = \ln(\frac{1}{9}) + \ln(x^2) = \ln(3^{-2}) + \ln(x^2) = -2\ln(3) + \ln(x^2)$.
Dus de translatie $(0, -2\ln(3))$ levert dezelfde beeldfiguur op.

8 **a** $f(x) = (x-3)^2 e^x = (x^2 - 6x + 9)e^x$ geeft $f'(x) = (2x-6) \cdot e^x + (x^2 - 6x + 9) \cdot e^x = (x^2 - 4x + 3)e^x$

$f'(x) = 0$ geeft $(x^2 - 4x + 3)e^x = 0$

$x^2 - 4x + 3 = 3$

$(x-1)(x-3) = 0$

$x = 1 \vee x = 3$

$f(1) = (1-3)^2 e^1 = 4e$ en $f(3) = (3-3)^2 e^3 = 0$

De toppen zijn $(1, 4e)$ en $(3, 0)$.

b $f'(x) = (x^2 - 4x + 3)e^x$ geeft $f''(x) = (2x-4) \cdot e^x + (x^2 - 4x + 3) \cdot e^x = (x^2 - 2x - 1)e^x$

$f''(x) = 0$ geeft $(x^2 - 2x - 1)e^x = 0$

$x^2 - 2x - 1 = 0$

$D = (-2^2) - 4 \cdot 1 \cdot -1 = 8$, dus $\sqrt{D} = \sqrt{8} = 2\sqrt{2}$

$x = \dfrac{2 + 2\sqrt{2}}{2} \vee x = \dfrac{2 - 2\sqrt{2}}{2}$

$x = 1 + \sqrt{2} \vee x = 1 - \sqrt{2}$

De x-coördinaten van de buigpunten zijn $x = 1 + \sqrt{2}$ en $x = 1 - \sqrt{2}$.

c Stel $x_A = a$, dan is $x_B = a + 3$.

$p = f(x_A) = f(x_B)$ geeft $(a-3)^2 e^a = a^2 e^{a+3}$

Voer in $y_1 = (x-3)^2 e^x$ en $y_2 = x^2 e^{x+3}$.

De optie snijpunt geeft $x = -0{,}861...$ en $y = 6{,}299...$ en $x = 0{,}547...$ en $y = 10{,}398...$

Dus $p \approx 6{,}30$.

9 **a** $f_{-2}(x) = x^{-2} \cdot \ln(x)$ geeft $f_{-2}'(x) = -2x^{-3} \cdot \ln(x) + x^{-2} \cdot \dfrac{1}{x} = \dfrac{-2\ln(x)}{x^3} + \dfrac{1}{x^2} \cdot \dfrac{1}{x} = \dfrac{-2\ln(x)}{x^3} + \dfrac{1}{x^3} = \dfrac{-2\ln(x) + 1}{x^3}$

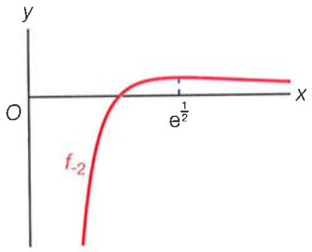
$f_{-2}'(x) = 0$ geeft $\dfrac{-2\ln(x) + 1}{x^3} = 0$

$-2\ln(x) + 1 = 0$

$-2\ln(x) = -1$

$\ln(x) = \frac{1}{2}$

$x = e^{\frac{1}{2}}$



max. is $f_{-2}(e^{\frac{1}{2}}) = (e^{\frac{1}{2}})^{-2} \cdot \frac{1}{2} = e^{-1} \cdot \frac{1}{2} = \dfrac{1}{2e}$

b $f_3(x) = x^3 \cdot \ln(x)$ geeft $f_3'(x) = 3x^2 \cdot \ln(x) + x^3 \cdot \dfrac{1}{x} = 3x^2 \ln(x) + x^2$

$f_3'(x) = 3x^2\ln(x) + x^2$ geeft $f_3''(x) = 6x \cdot \ln(x) + 3x^2 \cdot \dfrac{1}{x} + 2x = 6x\ln(x) + 3x + 2x = 6x\ln(x) + 5x$

$f_3''(x) = 0$ geeft $6x\ln(x) + 5x = 0$

$x(6\ln(x) + 5) = 0$

$x = 0 \;\vee\; 6\ln(x) + 5 = 0$

vold. niet $6\ln(x) = -5$

$\ln(x) = -\frac{5}{6}$

$x = e^{-\frac{5}{6}}$

De x-coördinaat van het buigpunt van de grafiek van f_3 is $x = e^{-\frac{5}{6}}$.

c $f_p(x) = x^p \cdot \ln(x)$ geeft $f_p'(x) = px^{p-1} \cdot \ln(x) + x^p \cdot \frac{1}{x} = px^{p-1}\ln(x) + x^p \cdot x^{-1} = px^{p-1}\ln(x) + x^{p-1}$

$f_p'(x) = 0$ geeft $px^{p-1}\ln(x) + x^{p-1} = 0$

$x^{p-1}(p\ln(x) + 1) = 0$

$x = 0 \vee p\ln(x) + 1 = 0$

vold. niet $\quad p\ln(x) = -1$

$\ln(x) = -\frac{1}{p}$

$x = e^{-\frac{1}{p}}$

$f_p(e^{-\frac{1}{p}}) = (e^{-\frac{1}{p}})^p \cdot -\frac{1}{p} = e^{-1} \cdot -\frac{1}{p} = -\frac{1}{pe}$

$y_{top} = \frac{1}{3}e$ geeft $-\frac{1}{pe} = \frac{1}{3}e$

$-\frac{1}{pe} = \frac{e}{3}$

$pe^2 = -3$

$p = \frac{-3}{e^2}$

10 **a** $f(x) = g(x)$ geeft $\ln(4x) = \ln\left(\frac{1}{x}\right)$

$4x = \frac{1}{x}$

$4x^2 = 1$

$x^2 = \frac{1}{4}$

$x = \frac{1}{2} \vee x = -\frac{1}{2}$

vold. niet

$f(\frac{1}{2}) = \ln(2)$ geeft $A(\frac{1}{2}, \ln(2))$

$f(x) = \ln(4x) = \ln(4) + \ln(x)$ geeft $f'(x) = \frac{1}{x}$

Stel $y = ax + b$ met $a = f'(\frac{1}{2}) = 2$.

$\left.\begin{array}{l} y = 2x + b \\ \text{door } A(\frac{1}{2}, \ln(2)) \end{array}\right\} 2 \cdot \frac{1}{2} + b = \ln(2)$

$b = \ln(2) - 1$

De lijn $y = 2x + \ln(2) - 1$ snijdt de y-as in $(0, \ln(2) - 1)$.

$g(x) = \ln\left(\frac{1}{x}\right) = \ln(x^{-1}) = -\ln(x)$ geeft $g'(x) = -\frac{1}{x}$

Stel $y = ax + b$ met $a = g'(\frac{1}{2}) = -2$.

$\left.\begin{array}{l} y = -2x + b \\ \text{door } A(\frac{1}{2}, \ln(2)) \end{array}\right\} -2 \cdot \frac{1}{2} + b = \ln(2)$

$b = \ln(2) + 1$

De lijn $y = -2x + \ln(2) + 1$ snijdt de y-as in $(0, \ln(2) + 1)$.

De lengte van het gevraagde lijnstuk is $\ln(2) + 1 - (\ln(2) - 1) = 2$.

b Stel $x_P = p$.

$PS = g(p) = \ln\left(\frac{1}{p}\right)$

$\left.\begin{array}{l} PQRS \text{ is een vierkant, dus } PQ = PS = \ln\left(\frac{1}{p}\right) \\ x_P = p \end{array}\right\} x_Q = p + \ln\left(\frac{1}{p}\right)$

$x_Q = p + \ln\left(\frac{1}{p}\right)$ geeft $QR = f\left(p + \ln\left(\frac{1}{p}\right)\right) = \ln\left(4\left(p + \ln\left(\frac{1}{p}\right)\right)\right) = \ln\left(4p + 4\ln\left(\frac{1}{p}\right)\right)$

$PQRS$ is een vierkant, dus $PS = QR$

$\ln\left(\frac{1}{p}\right) = \ln\left(4p + 4\ln\left(\frac{1}{p}\right)\right)$

$\frac{1}{p} = 4p + 4\ln\left(\frac{1}{p}\right)$

Voer in $y_1 = \frac{1}{x}$ en $y_2 = 4x + 4\ln\left(\frac{1}{x}\right)$.

De optie snijpunt geeft $x = 0{,}1065...$

Dus $x_P \approx 0{,}107$.

Bladzijde 228

11 **a** $p = 10\,\mathrm{e}^{-\frac{1}{2}t+1}$

$\mathrm{e}^{-\frac{1}{2}t+1} = \frac{1}{10}p$

$-\frac{1}{2}t + 1 = \ln(\frac{1}{10}p)$

$-\frac{1}{2}t = -1 + \ln(\frac{1}{10}p)$

$t = 2 - 2\ln(\frac{1}{10}p)$

$t = \ln(\mathrm{e}^2) + \ln((\frac{1}{10}p)^{-2})$

$t = \ln(\mathrm{e}^2) + \ln(100p^{-2})$

$t = \ln(100\mathrm{e}^2p^{-2})$

b $y = 3\ln\left(\frac{2}{x}\right) + 2\ln(5x)$

$y = 3(\ln(2) - \ln(x)) + 2(\ln(5) + \ln(x))$

$y = 3\ln(2) - 3\ln(x) + 2\ln(5) + 2\ln(x)$

$y = \ln(2^3) + \ln(5^2) - \ln(x)$

$\ln(x) = \ln(2^3 \cdot 5^2) - y$

$\ln(x) = \ln(200) + \ln(\mathrm{e}^{-y})$

$\ln(x) = \ln(200\,\mathrm{e}^{-y})$

$x = 200\,\mathrm{e}^{-y}$

c $y = \mathrm{e}^x + \mathrm{e}^{x+1} + \mathrm{e}^{x+2} + \mathrm{e}^{x+3}$

$\mathrm{e}^x + \mathrm{e}^{x+1} + \mathrm{e}^{x+2} + \mathrm{e}^{x+3} = y$

$\mathrm{e}^x + \mathrm{e}^x \cdot \mathrm{e} + \mathrm{e}^x \cdot \mathrm{e}^2 + \mathrm{e}^x \cdot \mathrm{e}^3 = y$

$\mathrm{e}^x(1 + \mathrm{e} + \mathrm{e}^2 + \mathrm{e}^3) = y$

$\mathrm{e}^x = \dfrac{y}{1 + \mathrm{e} + \mathrm{e}^2 + \mathrm{e}^3}$

$x = \ln\left(\dfrac{y}{1 + \mathrm{e} + \mathrm{e}^2 + \mathrm{e}^3}\right)$

$\dfrac{\mathrm{e}-1}{\mathrm{e}^4-1} = \dfrac{\mathrm{e}-1}{(\mathrm{e}^2+1)(\mathrm{e}^2-1)} = \dfrac{\mathrm{e}-1}{(\mathrm{e}^2+1)(\mathrm{e}+1)(\mathrm{e}-1)} = \dfrac{1}{(\mathrm{e}^2+1)(\mathrm{e}+1)} = \dfrac{1}{\mathrm{e}^3+\mathrm{e}^2+\mathrm{e}+1}$

Dus $x = \ln\left(\dfrac{y}{1 + \mathrm{e} + \mathrm{e}^2 + \mathrm{e}^3}\right)$ kan worden geschreven als $x = \ln\left(\dfrac{\mathrm{e}-1}{\mathrm{e}^4-1}y\right)$.

10 Meetkunde met vectoren

12 **a** Substitutie van $x = -1 + 5t$ en $y = -t$ in $3x + 2y = 10$ geeft $3(-1 + 5t) + 2 \cdot -t = 10$

$-3 + 15t - 2t = 10$

$13t = 13$

$t = 1$

$t = 1$ geeft $x = -1 + 5 = 4$ en $y = -1$, dus $S(4, -1)$.

b $m \perp k$, dus $\vec{n}_m = \vec{r}_k = \begin{pmatrix} 5 \\ -1 \end{pmatrix}$.

$\left.\begin{array}{l} m\colon 5x - y = c \\ \text{door } A(1, 3) \end{array}\right\} c = 5 \cdot 1 - 3 = 2$

Dus m: $5x - y = 2$.

c $\vec{n}_l = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$, dus $\vec{r}_l = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$.

$$\cos(\angle(k, l)) = \dfrac{\left|\begin{pmatrix} 5 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -3 \end{pmatrix}\right|}{\left|\begin{pmatrix} 5 \\ -1 \end{pmatrix}\right| \cdot \left|\begin{pmatrix} 2 \\ -3 \end{pmatrix}\right|} = \dfrac{|10 + 3|}{\sqrt{26} \cdot \sqrt{13}} = \dfrac{13}{13\sqrt{2}} = \tfrac{1}{2}\sqrt{2}$$

$\angle(k, l) = 45°$