11 Integraalrekening

Bladzijde 231

22 **a** $F(x) = x + \ln\left(\frac{x-1}{x+1}\right)$ geeft

$F'(x) = 1 + \frac{1}{\frac{x-1}{x+1}} \cdot \frac{(x+1)\cdot 1 - (x-1)\cdot 1}{(x+1)^2} = 1 + \frac{x+1}{x-1} \cdot \frac{x+1-x+1}{(x+1)^2} = 1 + \frac{2}{(x-1)(x+1)} = \frac{x^2-1+2}{x^2-1}$

$= \frac{x^2+1}{x^2-1}$

Dus $F'(x) = f(x)$ oftewel F is een primitieve van f.

b $F(x) = x\ln\left(\frac{x+1}{x-1}\right) + \ln(x^2-1)$ geeft

$F'(x) = 1 \cdot \ln\left(\frac{x+1}{x-1}\right) + x \cdot \frac{1}{\frac{x+1}{x-1}} \cdot \frac{(x-1)\cdot 1 - (x+1)\cdot 1}{(x-1)^2} + \frac{1}{x^2-1} \cdot 2x$

$= \ln\left(\frac{x+1}{x-1}\right) + x \cdot \frac{x-1}{x+1} \cdot \frac{x-1-x-1}{(x-1)^2} + \frac{2x}{x^2-1} = \ln\left(\frac{x+1}{x-1}\right) + x \cdot \frac{x-1}{x+1} \cdot \frac{-2}{(x-1)^2} + \frac{2x}{x^2-1}$

$= \ln\left(\frac{x+1}{x-1}\right) + x \cdot \frac{1}{x+1} \cdot \frac{-2}{x-1} + \frac{2x}{x^2-1} = \ln\left(\frac{x+1}{x-1}\right) - \frac{2x}{x^2-1} + \frac{2x}{x^2-1} = \ln\left(\frac{x+1}{x-1}\right)$

Dus $F'(x) = f(x)$ oftewel F is een primitieve van f.

c $F(x) = -x^3\cos(x) + 3x^2\sin(x) + 6x\cos(x) - 6\sin(x)$ geeft

$F'(x) = -3x^2 \cdot \cos(x) - x^3 \cdot -\sin(x) + 6x \cdot \sin(x) + 3x^2 \cdot \cos(x) + 6 \cdot \cos(x) + 6x \cdot -\sin(x) - 6\cos(x)$

$= -3x^2\cos(x) + x^3\sin(x) + 6x\sin(x) + 3x^2\cos(x) + 6\cos(x) - 6x\sin(x) - 6\cos(x)$

$= x^3\sin(x)$

Dus $F'(x) = f(x)$ oftewel F is een primitieve van f.

d $F(x) = (6x^2 - 2x - 4)\sqrt{x-1}$ geeft

$F'(x) = (12x-2) \cdot \sqrt{x-1} + (6x^2-2x-4) \cdot \frac{1}{2\sqrt{x-1}} = (12x-2)\sqrt{x-1} + \frac{3x^2-x-2}{\sqrt{x-1}}$

$= \frac{(12x-2)(x-1)}{\sqrt{x-1}} + \frac{3x^2-x-2}{\sqrt{x-1}} = \frac{12x^2-12x-2x+2+3x^2-x-2}{\sqrt{x-1}}$

$= \frac{15x^2-15x}{\sqrt{x-1}} = \frac{15x(x-1)}{\sqrt{x-1}} = 15x\sqrt{x-1}$

Dus $F'(x) = f(x)$ oftewel F is een primitieve van f.

23 **a** $f(x) = 7 \cdot \log(5x)$ geeft $F(x) = 7 \cdot \frac{1}{5} \cdot \frac{1}{\ln(10)} \cdot (5x\ln(5x) - 5x) + c = \frac{7(x\ln(5x) - x)}{\ln(10)} + c$

b $f(x) = e^{\frac{1}{2}x-1}$ geeft $F(x) = 2e^{\frac{1}{2}x-1} + c$

c $f(x) = 10\ln(2x-4)$ geeft

$F(x) = 10 \cdot \frac{1}{2}((2x-4)\ln(2x-4) - (2x-4)) + c = 5(2x-4)\ln(2x-4) - 5(2x-4) + c$

d $f(x) = \frac{x^4 - 6x^2\sqrt{x} + 8x}{x^3} = \frac{x^4}{x^3} - \frac{6x^{2\frac{1}{2}}}{x^3} + \frac{8x}{x^3} = x - 6x^{-\frac{1}{2}} + 8x^{-2}$ geeft

$F(x) = \frac{1}{2}x^2 - \frac{6}{\frac{1}{2}} \cdot x^{\frac{1}{2}} + \frac{8}{-1} \cdot x^{-1} + c = \frac{1}{2}x^2 - 12\sqrt{x} - \frac{8}{x} + c$

e $f(x) = (x^2+3)^2 = x^4 + 6x^2 + 9$ geeft $F(x) = \frac{1}{5}x^5 + 2x^3 + 9x + c$

f $f(x) = 3\sin(4x)$ geeft $F(x) = 3 \cdot -\frac{1}{4}\cos(4x) + c = -\frac{3}{4}\cos(4x) + c$

24 **a** $f(x) = \sqrt{6x+3} = (6x+3)^{\frac{1}{2}}$ geeft $F(x) = \frac{1}{6} \cdot \frac{2}{3}(6x+3)^{1\frac{1}{2}} + c = \frac{1}{9}(6x+3)\sqrt{6x+3} + c$

b $f(x) = \frac{10}{2x-1}$ geeft $F(x) = 10 \cdot \frac{1}{2}\ln|2x-1| + c = 5\ln|2x-1| + c$

c $f(x) = (3x-6)^{-2}$ geeft $F(x) = \frac{1}{3} \cdot -(3x-6)^{-1} + c = -\frac{1}{3}(3x-6)^{-1} + c$

d $f(x) = (2x+5)^{-1} = \frac{1}{2x+5}$ geeft $F(x) = \frac{1}{2}\ln|2x+5| + c$

e $f(x) = 10^{2x-3}$ geeft $F(x) = \frac{1}{2} \cdot \frac{10^{2x-3}}{\ln(10)} + c = \frac{10^{2x-3}}{2\ln(10)} + c$

f $f(x) = \frac{8^x - 1}{2^x} = \frac{8^x}{2^x} - \frac{1}{2^x} = 4^x - 2^{-x}$ geeft $F(x) = \frac{4^x}{\ln(4)} + \frac{2^{-x}}{\ln(2)} + c$

25 $F(x) = (ax^2 + bx + c)\,\mathrm{e}^x$ geeft
$F'(x) = (2ax + b) \cdot \mathrm{e}^x + (ax^2 + bx + c) \cdot \mathrm{e}^x = (ax^2 + (2a + b)x + b + c)\,\mathrm{e}^x$
Dit moet gelijk zijn aan $f(x) = (x^2 + 1)\,\mathrm{e}^x$, dus $a = 1$, $2a + b = 0$ en $b + c = 1$.

$$\left.\begin{array}{l} a = 1 \\ 2a + b = 0 \end{array}\right\} \begin{array}{l} 2 + b = 0 \\ b = -2 \end{array}$$

$$\left.\begin{array}{l} b = -2 \\ b + c = 1 \end{array}\right\} \begin{array}{l} -2 + c = 1 \\ c = 3 \end{array}$$

Dus $a = 1$, $b = -2$ en $c = 3$.

26 $f(x) = px$ geeft $x^3 - 3x = px$
$x^3 = px + 3x$
$x^3 = (p + 3)x$
$x = 0 \vee x^2 = p + 3$
$x = 0 \vee x = \sqrt{p+3} \vee x = -\sqrt{p+3}$
vold. niet

$$O(V) = \int_0^{\sqrt{p+3}} (px - f(x))\,\mathrm{d}x = \int_0^{\sqrt{p+3}} (px - (x^3 - 3x))\,\mathrm{d}x = \int_0^{\sqrt{p+3}} (-x^3 + (p+3)x)\,\mathrm{d}x$$

$$= \left[-\tfrac{1}{4}x^4 + (p+3) \cdot \tfrac{1}{2}x^2\right]_0^{\sqrt{p+3}} = -\tfrac{1}{4} \cdot (p+3)^2 + (p+3) \cdot \tfrac{1}{2} \cdot (p+3) - 0 = \tfrac{1}{4}(p+3)^2$$

$O(V) = 9$ geeft $\tfrac{1}{4}(p+3)^2 = 9$
$(p+3)^2 = 36$
$p + 3 = 6 \vee p + 3 = -6$
$p = 3 \vee p = -9$
vold. niet

Dus $p = 3$.

27 $O(ABCD) = 5 \cdot (2 - p) = 10 - 5p$
V is het vlakdeel dat wordt ingesloten door de grafiek van f en het lijnstuk CD.

$$O(V) = \int_0^5 (2 - f(x))\,\mathrm{d}x = \int_0^5 \left(2 - \tfrac{1}{2}x + 1 - \frac{3}{x+1}\right)\mathrm{d}x = \int_0^5 \left(3 - \tfrac{1}{2}x - \frac{3}{x+1}\right)\mathrm{d}x = \left[3x - \tfrac{1}{4}x^2 - 3\ln|x+1|\right]_0^5$$

$$= 15 - 6\tfrac{1}{4} - 3\ln(6) - (0 - 0 - 0)) = 8\tfrac{3}{4} - 3\ln(6)$$

$O(V) = \tfrac{1}{2}O(ABCD)$ geeft $8\tfrac{3}{4} - 3\ln(6) = 5 - 2\tfrac{1}{2}p$
$2\tfrac{1}{2}p = 3\ln(6) - 3\tfrac{3}{4}$
$p = 1\tfrac{1}{5}\ln(6) - 1\tfrac{1}{2}$

Bladzijde 232

28 **a** 54 km/uur = 15 m/s
Versnelling van de wielrenner constant a en beginsnelheid 15 geeft snelheid $v_w(t) = at + 15$.
$v_w(t) = at + 15$ geeft $s_w(t) = \tfrac{1}{2}at^2 + 15t + c$

$$\left.\begin{array}{l} s_w(t) = \tfrac{1}{2}at^2 + 15t + c \\ s_w(0) = 0 \end{array}\right\} s_w(t) = \tfrac{1}{2}at^2 + 15t$$

$v_w(t) = 0$ geeft $at + 15 = 0$
$at = -15$
$t = -\dfrac{15}{a}$

$s_w\left(-\frac{15}{a}\right) = 40$ geeft $\frac{1}{2}a \cdot \left(-\frac{15}{a}\right)^2 + 15 \cdot -\frac{15}{a} = 40$

$\frac{1}{2}a \cdot \frac{225}{a^2} - \frac{225}{a} = 40$

$\frac{112\frac{1}{2}}{a} - \frac{225}{a} = 40$

$-\frac{112\frac{1}{2}}{a} = 40$

$a = -\frac{112\frac{1}{2}}{40} = -2{,}8125$

Dus de versnelling van de wielrenner is $-2{,}8125$ m/s^2.

b $a = -2{,}5$, $v_w(0) = 15$ en $s_w(0) = 0$ geeft $v_w(t) = -2{,}5t + 15$ en $s_w(t) = -1{,}25t^2 + 15t$

6 km/uur $= 1\frac{2}{3}$ m/s

Voor de auto geldt $v_a(t) = 1\frac{2}{3}$.

$\left.\begin{array}{l} v_a(t) = 1\frac{2}{3} \text{ geeft } s_a(t) = 1\frac{2}{3}t + b \\ s_a(0) = 40 \end{array}\right\} s_a(t) = 1\frac{2}{3}t + 40$

$v_w(t) = v_a(t)$

$-2{,}5t + 15 = 1\frac{2}{3}$

$-2{,}5t = -13\frac{1}{3}$

$t = 5\frac{1}{3}$

$s_w(5\frac{1}{3}) = -1{,}25 \cdot (5\frac{1}{3})^2 + 15 \cdot 5\frac{1}{3} = 44\frac{4}{9}$

$s_a(5\frac{1}{3}) = 1\frac{2}{3} \cdot 5\frac{1}{3} + 40 = 48\frac{8}{9}$

Dus op het moment dat de wielrenner en de auto dezelfde snelheid hebben, rijdt de auto $48\frac{8}{9} - 44\frac{4}{9} = 4\frac{4}{9}$ meter voor de wielrenner.

De wielrenner kan dus op tijd stoppen.

29 **a** $O(V) = \int_0^1 (3^x + 1)\,dx = \left[\frac{3^x}{\ln(3)} + x\right]_0^1 = \frac{3}{\ln(3)} + 1 - \frac{1}{\ln(3)} = \frac{2}{\ln(3)} + 1$

b $I(L) = \pi\int_0^1 (3^x + 1)^2\,dx = \pi\int_0^1 (3^{2x} + 2 \cdot 3^x + 1)\,dx = \pi\left[\frac{1}{2} \cdot \frac{3^{2x}}{\ln(3)} + 2 \cdot \frac{3^x}{\ln(3)} + x\right]_0^1$

$= \pi\left(\frac{4\frac{1}{2}}{\ln(3)} + \frac{6}{\ln(3)} + 1 - \left(\frac{\frac{1}{2}}{\ln(3)} + \frac{2}{\ln(3)} + 0\right)\right) = \pi\left(\frac{4\frac{1}{2} + 6 - \frac{1}{2} - 2}{\ln(3)} + 1\right) = \frac{8\pi}{\ln(3)} + \pi$

c $y = 3^x + 1$ geeft $3^x + 1 = y$

$3^x = y - 1$

$x = {}^3\log(y - 1)$

$f(0) = 2$ en $f(1) = 4$

$I(M) = \pi \cdot 1^2 \cdot 4 - \pi\int_2^4 ({}^3\log(y - 1))^2\,dy \approx 9{,}89$

30 **a** $I = \pi\int_{\frac{1}{3}r}^{\frac{1}{2}r} y^2\,dx = \pi\int_{\frac{1}{3}r}^{\frac{1}{2}r} (r^2 - x^2)\,dx = \pi\left[r^2x - \frac{1}{3}x^3\right]_{\frac{1}{3}r}^{\frac{1}{2}r} = \pi(r^2 \cdot \frac{1}{2}r - \frac{1}{3} \cdot (\frac{1}{2}r)^3) - \pi(r^2 \cdot \frac{1}{3}r - \frac{1}{3} \cdot (\frac{1}{3}r)^3)$

$= \pi(\frac{1}{2}r^3 - \frac{1}{24}r^3) - \pi(\frac{1}{3}r^3 - \frac{1}{81}r^3) = \frac{11}{24}\pi r^3 - \frac{26}{81}\pi r^3 = \frac{89}{648}\pi r^3$

b $I_{bol} = \frac{4}{3}\pi r^3$

$I = \pi\int_{-pr}^{pr} y^2\,dx = \pi\int_{-pr}^{pr} (r^2 - x^2)\,dx = \pi\left[r^2x - \frac{1}{3}x^3\right]_{-pr}^{pr} = \pi(r^2 \cdot pr - \frac{1}{3}(pr)^3) - \pi(r^2 \cdot -pr - \frac{1}{3}(-pr)^3)$

$= \pi(pr^3 - \frac{1}{3}p^3r^3) + \pi(pr^3 - \frac{1}{3}p^3r^3) = \pi(2pr^3 - \frac{2}{3}p^3r^3) = (2p - \frac{2}{3}p^3)\pi r^3$

$I = \frac{1}{2}I_{bol}$ geeft $(2p - \frac{2}{3}p^3)\pi r^3 = \frac{2}{3}\pi r^3$

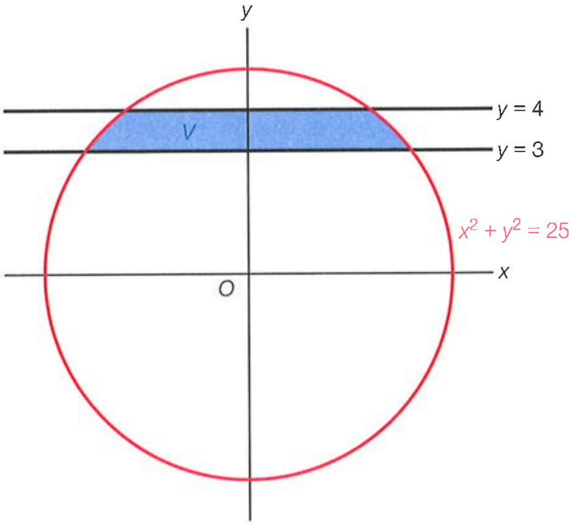
$2p - \frac{2}{3}p^3 = \frac{2}{3}$

Voer in $y_1 = 2x - \frac{2}{3}x^3$ en $y_2 = \frac{2}{3}$.

De optie snijpunt geeft $x = 0{,}347...$

Dus $p \approx 0{,}35$.

31



Substitutie van $y = 3$ in $x^2 + y^2 = 25$ geeft $x^2 + 9 = 25$

$x^2 = 16$

$x = 4 \vee x = -4$

Substitutie van $y = 4$ in $x^2 + y^2 = 25$ geeft $x^2 + 16 = 25$

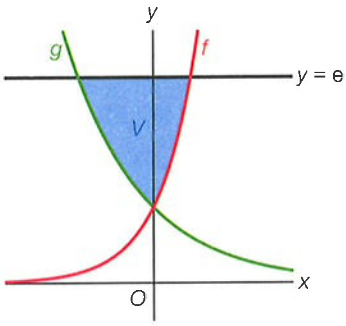
$x^2 = 9$

$x = 3 \vee x = -3$

$$I(L) = \pi\int_{-4}^{-3} y^2\,dx + \pi\int_{-3}^{3} 4^2\,dx + \pi\int_{3}^{4} y^2\,dx - \pi\int_{-4}^{4} 3^2\,dx$$
$$= \pi\int_{-4}^{-3} (25 - x^2)\,dx + \pi\left[16x\right]_{-3}^{3} + \pi\int_{3}^{4}(25 - x^2)\,dx - \pi\left[9x\right]_{-4}^{4}$$
$$= \pi\left[25x - \tfrac{1}{3}x^3\right]_{-4}^{-3} + \pi(48 - -48) + \pi\left[25x - \tfrac{1}{3}x^3\right]_{3}^{4} - \pi(36 - -36)$$
$$= \pi(-75 + 9) - \pi(-100 + \tfrac{64}{3}) + 96\pi + \pi(100 - \tfrac{64}{3}) - \pi(75 - 9) - 72\pi$$
$$= -66\pi + 78\tfrac{2}{3}\pi + 96\pi + 78\tfrac{2}{3}\pi - 66\pi - 72\pi = 49\tfrac{1}{3}\pi$$

Bladzijde 233

32 **a**



$e^{2x} = e$	$e^{-x} = e$	$e^{2x} = e^{-x}$
$2x = 1$	$-x = 1$	$2x = -x$
$x = \frac{1}{2}$	$x = -1$	$x = 0$

e omlaag schuiven geeft

$$I(L) = \pi\int_{-1}^{0}(e^{-x} - e)^2\,dx + \pi\int_{0}^{\frac{1}{2}}(e^{2x} - e)^2\,dx = \pi\int_{-1}^{0}(e^{-2x} - 2\,e^{-x+1} + e^2)\,dx + \pi\int_{0}^{\frac{1}{2}}(e^{4x} - 2\,e^{2x+1} + e^2)\,dx$$
$$= \pi\left[-\tfrac{1}{2}e^{-2x} + 2\,e^{-x+1} + e^2x\right]_{-1}^{0} + \pi\left[\tfrac{1}{4}e^{4x} - e^{2x+1} + e^2x\right]_{0}^{\frac{1}{2}}$$
$$= \pi(-\tfrac{1}{2} + 2e) - \pi(-\tfrac{1}{2}e^2 + 2e^2 - e^2) + \pi(\tfrac{1}{4}e^2 - e^2 + \tfrac{1}{2}e^2) - \pi(\tfrac{1}{4} - e) = (-\tfrac{3}{4}e^2 + 3e - \tfrac{3}{4})\pi$$

b $\int_{-1}^{0} g(x)\,dx = \int_{-1}^{0} e^{-x}\,dx = [-e^{-x}]_{-1}^{0} = -1 + e$

$\int_{0}^{\frac{1}{2}} f(x)\,dx = \int_{0}^{\frac{1}{2}} e^{2x}\,dx = [\frac{1}{2}e^{2x}]_{0}^{\frac{1}{2}} = \frac{1}{2}e - \frac{1}{2}$

Dus $O(W) = -1 + e + \frac{1}{2}e - \frac{1}{2} = 1\frac{1}{2}e - 1\frac{1}{2}$.

$\int_{-1}^{\ln(\frac{4}{e+3})} e^{-x}\,dx = \left[-e^{-x}\right]_{-1}^{\ln(\frac{4}{e+3})} = -e^{-\ln(\frac{4}{e+3})} + e = -e^{\ln(\frac{e+3}{4})} + e = -\frac{e+3}{4} + e = -\frac{1}{4}e - \frac{3}{4} + e = \frac{3}{4}e - \frac{3}{4}$

$\frac{1}{2}O(W) = \frac{1}{2}(1\frac{1}{2}e - 1\frac{1}{2}) = \frac{3}{4}e - \frac{3}{4}$

Dus de lijn $x = \ln\left(\frac{4}{e+3}\right)$ verdeelt W in twee delen met gelijke oppervlakte.

12 Goniometrische formules

33 **a** $\sin(x + \frac{1}{3}\pi) = -\cos(x - \frac{2}{3}\pi)$

$\cos(x - \frac{1}{6}\pi) = \cos(x + \frac{1}{3}\pi)$

$x - \frac{1}{6}\pi = x + \frac{1}{3}\pi + k \cdot 2\pi \vee x - \frac{1}{6}\pi = -x - \frac{1}{3}\pi + k \cdot 2\pi$

geen oplossing $\qquad 2x = -\frac{1}{6}\pi + k \cdot 2\pi$

$x = -\frac{1}{12}\pi + k \cdot \pi$

b $2\sin(x)\cos(x) = \cos(x + \frac{1}{6}\pi)$

$\sin(2x) = \sin(x + \frac{2}{3}\pi)$

$2x = x + \frac{2}{3}\pi + k \cdot 2\pi \vee 2x = \pi - x - \frac{2}{3}\pi + k \cdot 2\pi$

$x = \frac{2}{3}\pi + k \cdot 2\pi \vee 3x = \frac{1}{3}\pi + k \cdot 2\pi$

$x = \frac{2}{3}\pi + k \cdot 2\pi \vee x = \frac{1}{9}\pi + k \cdot \frac{2}{3}\pi$

c $\cos(x + \frac{1}{2}\pi) \cdot \cos(x) = \frac{1}{4}\sqrt{2}$

$\cos(x + \frac{1}{2}\pi) \cdot \sin(x + \frac{1}{2}\pi) = \frac{1}{4}\sqrt{2}$

$\frac{1}{2}\sin(2x + \pi) = \frac{1}{4}\sqrt{2}$

$\sin(2x + \pi) = \frac{1}{2}\sqrt{2}$

$2x + \pi = \frac{1}{4}\pi + k \cdot 2\pi \vee 2x + \pi = \frac{3}{4}\pi + k \cdot 2\pi$

$2x = -\frac{3}{4}\pi + k \cdot 2\pi \vee 2x = -\frac{1}{4}\pi + k \cdot 2\pi$

$x = -\frac{3}{8}\pi + k \cdot \pi \vee x = -\frac{1}{8}\pi + k \cdot \pi$

d $\sqrt{3} \cdot \sin(2x) + \cos(2x) = 1$

$2\sqrt{3}\sin(x)\cos(x) + 1 - 2\sin^2(x) = 1$

$2\sqrt{3}\sin(x)\cos(x) - 2\sin^2(x) = 0$

$2\sin(x)(\sqrt{3} \cdot \cos(x) - \sin(x)) = 0$

$2\sin(x) = 0 \vee \sqrt{3} \cdot \cos(x) - \sin(x) = 0$

$\sin(x) = 0 \vee \sin(x) = \sqrt{3} \cdot \cos(x)$

$x = k \cdot \pi \vee \frac{\sin(x)}{\cos(x)} = \sqrt{3}$

$x = k \cdot \pi \vee \tan(x) = \sqrt{3}$

$x = k \cdot \pi \vee x = \frac{1}{3}\pi + k \cdot \pi$

34 **a** $f(x) = (2 - \sin(x))^2 = 4 - 4\sin(x) + \sin^2(x) = 4 - 4\sin(x) + \frac{1}{2} - \frac{1}{2}\cos(2x)$

$= 4\frac{1}{2} - 4\sin(x) - \frac{1}{2}\cos(2x)$

$F(x) = 4\frac{1}{2}x + 4\cos(x) - \frac{1}{4}\sin(2x) + c$

b $g(x) = \sin(3x)\cos(\frac{1}{2}x) + \cos(3x)\sin(\frac{1}{2}x) = \sin(3\frac{1}{2}x)$

$G(x) = -\frac{2}{7}\cos(3\frac{1}{2}x) + c$

c $h(x) = 4\cos^4(x) - \cos^2(2x) = (2\cos^2(x))^2 - \cos^2(2x) = (1 + \cos(2x))^2 - \cos^2(2x)$

$= 1 + 2\cos(2x) + \cos^2(2x) - \cos^2(2x) = 1 + 2\cos(2x)$

$H(x) = x + \sin(2x) + c$